2024-04-28 — Fourier crashcourse

$\mathcal{L}^p(\Omega;\mu)$ space: $\mathcal{L}^p(\Omega;\mu) = \{ f: \Omega \to \mathbb{R}$ (measurably) s.t. $\int_\Omega |f|^p d\mu < \infty \}$ $(\infty > p \geq 1)$

1 else if $p=\infty$ $\mathcal{L}^\infty(\Omega,\mu) = \{ f: \Omega \to \mathbb{R}$ s.t. $|f(\omega)| \leq C$ μ-a.e.$\}$

Hilbert space $\equiv \mathcal{L}^2(\Omega;\mu)$ | $\mathcal{C}^m$ space: continuous fns, deriv order m are contin.

$\leftarrow$ integrable functions

FT def: $f \in \mathcal{L}^1(\mathbb{R}^d)$ d: dimension of space

2 $\mathcal{F}[f]: \mathbb{R}^d \longmapsto \mathbb{C}$ Lebesgue integral

def by $\boxed{\mathcal{F}[f](\vec{k}) = \int_{\mathbb{R}^d} f(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}\, dx_1 \cdots dx_d}$

Props: Linearity $\mathcal{F}[\alpha f + \beta g] = \alpha \mathcal{F}[f] + \beta \mathcal{F}[g]$

Continuity: $\mathcal{F}[f] \in \mathcal{C}(\mathbb{R}^d)$ and $\|\mathcal{F}[f]\|_\infty \leq \|f\|_1$ Bounded

Shift: $\mathcal{F}[f(\vec{x}-\vec{\alpha})] = e^{-i\vec{k}\cdot\vec{\alpha}}\mathcal{F}[f]$

⚠ bad notation, but we understand... 3

Inv shift: $\mathcal{F}[e^{i\vec{\alpha}\cdot\vec{x}} f(\vec{x})](\vec{k}) = \mathcal{F}[f](\vec{k}-\vec{\alpha})$

Scale: $\mathcal{F}[f(\frac{\vec{x}}{\lambda})](\vec{k}) = \lambda^d \mathcal{F}[f](\lambda\vec{k})$

Hermitian complex conj: $\mathcal{F}[f^*(\vec{x})] = (-1)^d \mathcal{F}[f(-\vec{x})]^*$

~~Derivative of FT: if $\vec{x}^{\alpha} f(\vec{x}) \in \mathcal{L}^1(\mathbb{R}^d) \Rightarrow \mathcal{F}[f] \in \mathcal{C}^m$ and $\partial^\alpha \mathcal{F}[f] = (-i)^{|\alpha|} \mathcal{F}[x^\alpha f(x)]$~~

Application to ODEs: FTs of derivatives:

~~Let $f \in \mathcal{L}^1(\mathbb{R}^d) \cap \mathcal{C}^m(\mathbb{R}^d)$, $m > 0$~~

~~$\uparrow$ continuous + m'th derivatives cont.~~

~~and $\partial^\alpha f \in \mathcal{L}^1(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}^d$ multi-indices, $|\alpha| \leq m$~~

~~$$\partial^\alpha := \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1}(\partial x_2)^{\alpha_2}\cdots(\partial x_d)^{\alpha_d}} \qquad |\alpha| = \sum_{\mu=1}^{d} \alpha_\mu$$~~

~~$$\Rightarrow \mathcal{F}[\partial^\alpha f](\vec{k}) = i^{|\alpha|}\, \underbrace{k^{\alpha}}_{=\prod_{\mu=1}^{d} k_\mu^{\alpha_\mu}} \mathcal{F}[f](\vec{k})$$~~

Examples (4)

sym, $\det A > 0$ (pos-def)

- Gaussian integrals: $\mathcal{F}\left[e^{-\frac{1}{2}\underbrace{\langle\vec{x}|A|\vec{x}\rangle}_{x^TAx}}\right](\vec{k}) = \sqrt{\frac{(2\pi)^d}{\det A}}\, e^{\frac{1}{2}\langle\vec{k}|A^{-1}|\vec{k}\rangle}$

- Rect function: $\mathcal{F}[\mathbb{1}_{[-R,R]}] = \frac{2\sin(Rk)}{k}$ $(d=1)$

  ⚠ $\mathcal{F}[\mathbb{1}_{[-R,R]}] \notin \mathcal{L}^1(\mathbb{R}) \Rightarrow$ FTs of integrable functions are not always integ.

  → link with uncertainty principle

- Exponential decay (↔ link with calculus of residues)

$$\mathcal{F}[e^{-\alpha|x|}] = \frac{2\alpha}{\alpha^2+k^2}$$

- Integral

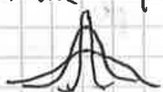
$$\mathcal{F}\left[\int_{-\infty}^{x} f(s)\,ds\right](k) = \frac{\mathcal{F}[f](k)}{ik} + \pi\mathcal{F}[f](k=0)\,\delta(k)$$

- Dirac delta

is a "distribution", as in $\delta[f] = f(0)$, often written as a limit of "pointy" fns:

$$\lim_{n\to\infty}\int_{\mathbb{R}} f(x)\delta_n(x)\,dx = \int_{\mathbb{R}} f(x)\delta(x)\,dx = f(0)$$

examples: gaussian $\delta_n = \frac{n}{\sqrt{2\pi}} e^{-\frac{n^2x^2}{2}}$



sinc: $\delta_n(x) = \frac{\sin(nx)}{\pi x}$

$$\mathcal{F}[\delta(x-x_0)](k) = e^{-ikx_0} \quad \left(\Leftrightarrow \delta(x-x_0) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ik(x-x_0)}\,dk\right)$$

$$\mathcal{F}[1](k) = \int_{\mathbb{R}} e^{-ikx} = \lim_{n\to\infty} 2\pi\delta_n(-k)$$

$$\left(\text{i.e.: } \delta_n(k) = \frac{1}{2\pi}\int_{-n}^{n} e^{ikx}\,dx\right)$$

⚠ δ_n is not "sym", it depends on the fn it is applied to

Inverse FT

Let $f \in \mathcal{L}^1(\mathbb{R}^d)$ st $\mathcal{F}[f] \in \mathcal{L}^1(\mathbb{R}^d)$

Then $\mathcal{F}^{-1}[\mathcal{F}[f]] = f$ a.e. in $\mathbb{R}^d$,

and if additionally $f \in \mathcal{C}(\mathbb{R}^d)$, then equality everywhere

$$\boxed{\mathcal{F}^{-1}[g](\vec{x}) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} g(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}\, d\vec{k}}$$

— $\mathcal{F}[f]$ is not always $\mathcal{L}^1(\mathbb{R}^d)$.

What are conditions that make this true?

- if $f \in \mathcal{L}^1(\mathbb{R}^d) \cap \mathcal{C}^m(\mathbb{R}^d)$ with $m > d$ st $\partial^\alpha f \in \mathcal{L}^1(\mathbb{R}^d)$ $\forall \alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq m$, then $\mathcal{F}[f] \in \mathcal{L}^1(\mathbb{R}^d)$

(note that if $f \in \mathcal{C}(\mathbb{R}^d)$ is decay poly, $f < \frac{cst}{\|x\|^m}$, then $f \in \mathcal{L}^1$)

— Note: we can go further in defining a space $\mathcal{S}$ of functions that is invariant under FT, ie $\mathcal{F}[\mathcal{S}(\mathbb{R}^d)] = \mathcal{S}(\mathbb{R}^d)$

$\mathcal{F}$ is bijective on functions in $\mathcal{S}(\mathbb{R}^d)$ $\left(\sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < \infty\right)$

"$\mathcal{S}(\mathbb{R}^d)$ is a space of fns that are smooth and decay rapidly

- Plancherel thm: $\boxed{(2\pi)^{-d} \langle \mathcal{F}[f], \mathcal{F}[g] \rangle = \langle f, g \rangle = \int \overline{f(x)}\, g(x)\, dx}$

— Note: FT can be extended to to fns in $\mathcal{L}^2(\mathbb{R}^d)$ st Plancherel thm holds.

Not all properties of FT carry over since $\mathcal{L}^1 \neq \mathcal{L}^2$, but do carry if $f \in \mathcal{L}^1(\mathbb{R}^d) \cap \mathcal{L}^2(\mathbb{R}^d)$

"$\hat{\delta}_0$"

— Note: FT extended to distribs, eg. Dirac $T(\varphi) := \varphi(0)$

FT and derivatives (& application to electromag)

Let $f \in \mathcal{L}^1(\mathbb{R}^d) \cap \mathcal{C}^m(\mathbb{R}^d)$, $m > 0$, st $\partial^\alpha f \in \mathcal{L}^1(\mathbb{R}^d)$ forall $\alpha \in \mathbb{N}_0^d$ st $|\alpha| \leq m$

6

$$\boxed{\mathcal{F}[\partial^\alpha f](\vec{k}) = i^{|\alpha|} k^\alpha \mathcal{F}[f](\vec{k})}$$

$\partial^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \cdots (\partial x_d)^{\alpha_d}}$, $|\alpha| = \sum_{i=1}^d \alpha_i$, $k^\alpha = \prod_{i=1}^d k_i^{\alpha_i}$

(Derivatives of FTs omitted — but can easily be recovered)

In practice (electromagnetism)

11

$E(\vec{x}, t)$ (electric field, any scalar component)

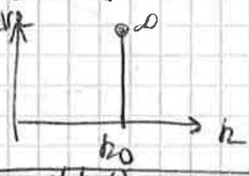
$$\tilde{E}(\vec{k}, \omega) = \int_{\mathbb{R}^3} d^3x \int_{\mathbb{R}} dt \, E(\vec{x}, t) e^{-i(\vec{k} \cdot \vec{x} - \omega t)}$$

conjugate for

why? Plane wave $E(\vec{x}, t) = e^{i(\vec{k}_0 \vec{x} - \omega_0 t)}$

$$\Rightarrow \tilde{E}(\vec{k}, \omega) = \int_{\mathbb{R}^3} d^3x \int_{\mathbb{R}} dt \, e^{-i((\vec{k} - \vec{k}_0)\vec{x} - (\omega - \omega_0)t)}$$

$$= \delta(\vec{k}_0 - \vec{k}) \, \delta(\omega - \omega_0)$$



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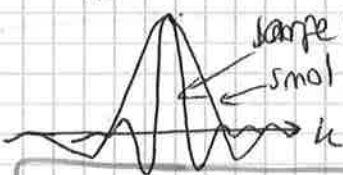
uncertainty principle

In general, examine a FT of $\mathrm{rect}_{[-R,R]} \cdot e^{i(k_0 x)}$

$$\mathcal{F}[\mathbb{1}_{-R,R} e^{ik_0 x}] = \mathcal{F}[\mathbb{1}_{-R,R}] * \mathcal{F}[e^{ik_0 x}] = \frac{2\sin(Rk)}{k} * 2\pi\delta(k_0 - k) = \frac{4\pi \sin(R(k-k_0))}{k - k_0}$$



$\Rightarrow$ uncertainty principle: 8

| larger t $\longrightarrow$ smaller freq width |

Heisenberg uncert: $\left(\int (x - x_0)^2 |f(x)|^2 dx\right)\left(\int (k - k_0)^2 |\mathcal{F}[f](k)|^2\right) \geq \frac{\pi}{2} \left(\int |f(x)|^2 dx\right)^2$

convolution theorem

$f, g \in L^1(\mathbb{R}^d)$; $f*g : \mathbb{R}^d \to \mathbb{R}$ de by

$$\boxed{(f*g)(\vec{x}) = \int_{\mathbb{R}^d} f(\vec{y})\, g(\vec{x}-\vec{y})\, d\vec{y}}$$

7

Props of convolutions

- $(f*g) \in \mathcal{L}^1(\mathbb{R}^d)$ (conserves $\mathcal{L}^1$-integ)
- Commut : $f*g = g*f$
- Associative: $f*(g*h) = (f*g)*h$
- Linear: $f*(g+h) = (f*g)+(f*h)$

$$\boxed{\begin{aligned} \mathcal{F}[f*g] &= \mathcal{F}[f]\cdot\mathcal{F}[g] \\ \mathcal{F}[f\cdot g] &= \mathcal{F}[f]*\mathcal{F}[g] \end{aligned}}$$

~~Examples: heat equation in $\mathbb{R}^d$~~

application of convolution theorem

Fourier series (19) (& link with FT)

Let $f(x)$ be a periodic fn with period P: $f(x+P) = f(x)$

complex form of fourier series: (usually f periodic on $[-L, L]$, here $P = 2L$)

$$\boxed{f(x) = \sum_{n \in \mathbb{Z}} c_n \underbrace{e^{ix\frac{2\pi n}{P}}}_{\text{ONB of functions!}}, \quad c_n = \frac{1}{P} \int_{\text{period}} f(x) e^{-ix\frac{2\pi n}{P}} dx}$$

$\Leftrightarrow$ "as $n \to \infty$, $P \to \infty$" ("periodic with period P")

$$f(x) = \frac{1}{(2\pi)^{(d)}} \int_{\mathbb{R}} c(k) e^{ikx} dk, \quad c(k) = \int_{\mathbb{R}} f(x) e^{-ikx} dx$$

- ~~Truncated~~ other forms and equivalences (see wikipedia)

sin-cos: $f(x) = \sum_{n=0}^{\infty} \left(a_n \cos(2\pi \frac{n}{P} x) + b_n \sin(2\pi \frac{n}{P} x)\right)$

ampl-phase $f(x) = d_0 + \sum_{n=1}^{\infty} d_n \cos(2\pi \frac{n}{P} x - \varphi_n)$

equiv: $c_n = \begin{cases} a_0 & n=0 \\ \frac{1}{2}(a_n - ib_n) & n>0 \\ \overline{c_{|n|}} & n<0 \end{cases} \Leftrightarrow \begin{array}{l} a_0 = c_0 \\ a_n = c_n + c_{-n} \\ b_n = i(c_n - c_{-n}) \end{array}$

[Note: if $f(x)$ is real, then $\overline{c_n} = c_{-n}$]

Note: if $f(x) = f(-x) \Rightarrow b_n = 0$ (cos-series)
$f(x) = -f(-x) \Rightarrow a_n = 0$ (sin-series)

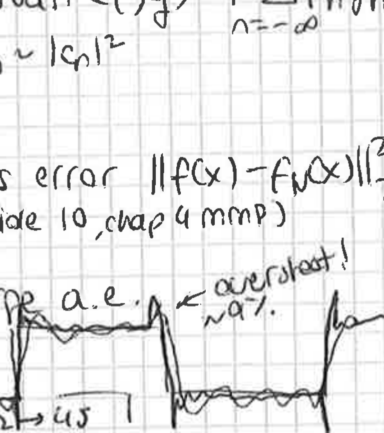
- many props carry over, eg Parseval: $\langle f, g \rangle = P \sum_{n=-\infty}^{\infty} \overline{f_n} g_n$
  ↳ spectral density is discrete: $c_n \sim |c_n|^2$

- Truncated Fourier series:

$f_N(x) = \sum_{n=-N}^{N} c_n e^{i2\pi \frac{nx}{P}}$. F.S. minimizes error $\|f(x) - f_N(x)\|_2^2$
(proof: slide 10, chap 4 mmp)

- Gibbs phenomenon: F.S. converge a.e.

one case where not is at discont.
(see wiki)



exercise: wave ~~heat~~ eq, 2D mmp slide 42 → 45

Energy spectrum and Parseval's / Plancherel's thm

Plancherel / Parseval's theorem / idenity

$$\boxed{\langle \mathcal{F}[f], \mathcal{F}[g] \rangle = (2\pi)^d \langle f, g \rangle}$$

may change depending on FT def

where $\langle f, g \rangle := \int_{\mathbb{R}^d} \overline{f(x)} g(x) dx$

case $f = g$: $\|\mathcal{F}[f]\|_2^2 = (2\pi)^d \|f\|_2^2$

— spectral energy density $\varepsilon \propto |\mathcal{F}[f]|^2$

why? energy stored in electric/mag fields

$$\varepsilon_{tot}(t) = \int_{\mathbb{R}^3 \text{ or a volume}} \underbrace{\frac{\epsilon_0 |\vec{E}(\vec{x},t)|^2}{2} + \frac{|\vec{B}(\vec{x},t)|^2}{2\mu_0}}_{\varepsilon(t),\text{ energy density}} d\vec{x}$$

electrostatic case, and $\vec{B} = 0$

$$\varepsilon(t) = \frac{\epsilon_0}{2} |E(x,t)|^2$$

so:

$$\varepsilon_{tot}(t) = \frac{\epsilon_0}{2} \iiint |E(x,t)|^2 dx = \frac{\epsilon_0}{2} \cdot \frac{1}{(2\pi)^3} \iiint \boxed{\underbrace{|\mathcal{F}[E](k,t)|^2}_{\propto \varepsilon(t)}} d\vec{k}$$

→ tells us what energies are associated with what wavelengths

Example: electromagnetism (11)

include example of plane wave, different definition in EM.

- Derivation of dispersion relation, starting from Maxwell's equations

$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ (Poisson) $\qquad \nabla\times E = -\frac{\partial B}{\partial t}$ induction

$\nabla\cdot B = 0$ (no mag [redacted] $\qquad \nabla\times B = \mu_0 J + \mu_0\epsilon_0\frac{\partial E}{\partial t}$ (induction)

$\Rightarrow$ waveeq: $\vec{\nabla}^2 E - \frac{1}{c^2}\partial_{tt}^2 E = \mu_0\partial_t J + \frac{1}{\epsilon_0}\vec{\nabla}\rho$, $\frac{1}{c^2} = \epsilon_0\mu_0$

alt form: $\vec{\nabla}\times(\vec{\nabla}\times E) = -\mu_0\partial_t J - \frac{1}{c^2}\partial_{tt}^2 E$

To Fourier: ~~[illegible]~~

with electromag defs: $\boxed{\partial_t \to -i\omega,\ \vec{\nabla}\to i\vec{k},\ \vec{\nabla}^2\to -k^2}$

⊕ constituent law: Ohm $\tilde{J}(\omega,\vec{k}) = \underbrace{\tilde{\sigma}(\omega,\vec{k})}_{\text{conduct tensor}}\tilde{E}(\omega,\vec{k})$

$\Rightarrow -\frac{c^2}{\omega^2}\underbrace{k\times(k\times\tilde{E})}_{[k_ik_j]} = i\omega\mu_0\tilde{J} + \frac{\omega^2}{c^2}\tilde{E} = \underbrace{\left(\frac{i}{\epsilon_0\omega}\tilde{\sigma} + \mathbb{1}\right)}_{\text{dielectric tensor } \tilde{\epsilon}}\tilde{E}$ (recall: $D = \epsilon E = \epsilon_0\epsilon_r E$
$= \epsilon_0(1+\chi_e)E$
$= \epsilon_0 E + \underbrace{\epsilon_0\chi_e E}_{\text{polariz. } P}$ is a fn of $\omega, \vec{k}$

$k^2\left(\frac{\vec{k}\vec{k}}{k^2} - \mathbb{1}\right)\tilde{E}$

$\Rightarrow \boxed{\det\left\{\underbrace{\frac{k^2c^2}{\omega^2}}_{=N^2 \text{ index of refraction}}\underbrace{\left[\frac{\vec{k}\vec{k}}{k^2} - \mathbb{1}\right]}_{=-P_{\perp\vec{k}}} + \tilde{\epsilon}\right\} = 0}$

$= -P_{\perp\vec{k}}$, projection on a plane orthogonal to $\vec{k}$ (wiki: vector proj)

$\rightarrow$ only combinations of $\vec{k}, \omega$ that satisfy dispersion rel can propagate.

Example: heat equation ; example of application of $\mathcal{F}[\partial^\alpha f]$ and convolution

$$\begin{cases} \vec{\nabla}^2 u = \frac{1}{D}\partial_t u, & D = \kappa^2,\ \kappa \text{ thermal conductivity},\ D \text{ diffusion coef.} \\ \text{initial cond} \quad u(0,x) = u_0(x), & x \in \mathbb{R}^d,\ t \in [0,\infty) \end{cases}$$

Ana IV Lecture, p. 44.

— other fun exercises to solve with FT

Poisson eq: $\vec{\nabla}^2 \varphi(x) = \frac{\rho(x)}{\epsilon_0}$, if $\rho(x) = 0$: Laplace eq.

Wave eq: $\vec{\nabla}^2 \psi - \frac{1}{u^2}\partial_t^2 \psi = 0$, $\psi(x,t)$

Heat eq: $\vec{\nabla}^2 \psi - \frac{1}{\kappa^2}\partial_t \psi = 0$ $\psi(x,t)$

Schrödinger: $\left[\frac{\hbar^2}{2m}\vec{\nabla}^2 - V(x,t)\right]\psi(x,t) + i\hbar\,\partial_t \psi(x,t) = 0$

Example: FT and complex analysis (13)

Recap: complex analysis

Laurent series: $f(z) = \sum_{n\in\mathbb{Z}} a_n (z-c)^n$

Residue a_{-1} of a pole of order k in $c = z_0$: $a_{-1} = \lim_{z\to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (z-z_0)^k f(z)$

Res thm: $\oint_\gamma f(z)dz = 2\pi i \sum_{c\in \text{singularities}} \text{Res}(f,c)\, \underbrace{\nu(\gamma,c)}_{\text{winding number of } \gamma \text{ around pole } c}$

App: improper integrals: $\int_{\mathbb{R}} dx\, f(x) = 2\pi i \sum_{s,\ \text{Im}(s)>0} \text{Res}(f,s)$, S singularities

" : Fourier tfs:

$$\boxed{\int_{\mathbb{R}} f(x) e^{i\omega x} dx = \pm 2\pi i \sum_{s\in S\cap\{\text{Im}(z)\gtrless 0\}} \text{Res}(f e^{i\omega z}, s); \quad \text{if } \omega \gtrless 0}$$

Example: FT of $\frac{2\alpha}{\alpha^2+x^2}$ see MMP chap 5, slide 23-25

FT of Gaussian H10.4.c, Ana IV.

Related to FT: Laplace transform

14

$$\mathcal{L}[f](s) = \int_0^\infty f(t)\, e^{-st}\, dt = \hat{f}(s)$$

$$f(t) = \mathcal{L}^{-1}[\hat{f}](t) = \frac{1}{2\pi i} \int_{p_0 - i\infty}^{p_0 + i\infty} \hat{f}(s)\, e^{st}\, ds,$$ p_0 st $\mathrm{Re}[p_0] > \max(\mathrm{Re}(\text{poles}))$

$$\left.\begin{aligned} \mathcal{L}[\partial_t f](s) &= s\hat{f}(s) - f(t=0) \\ \mathcal{L}[\partial_{tt} f](s) &= s^2 \hat{f}(s) - s f(t=0) - \partial_t f(t=0) \end{aligned}\right\} \text{differntl.}$$

$$\mathcal{L}[(f * g)](s) = \hat{f}(s)\, \hat{g}(s) \quad \} \text{ conv. thm}$$

$$\left.\begin{aligned} \mathcal{L}[e^{\alpha t} f(t)] &= \hat{f}(s - \alpha) \\ \mathcal{L}[\delta(t - t_0)] &= e^{-t_0 s} \\ \mathcal{L}[t^n f(t)](s) &= (-1)^n \partial^n \hat{f}(s) \end{aligned}\right\} \text{common fns}$$

15

Numerical implementations: FFT & how-to numpy

& minimal detectable frequency ([r]fft real signal)

numpy.fft.rfftfreq(len(t), d=np.diff(t)[0])

$\sqrt{E}$ = np.abs(np.fft.rfft(θ - $\langle\theta\rangle$))

~ powerspectrum

remove the mean!

$$\boxed{\Delta\text{freq} = \frac{1}{\Delta t \cdot (\text{num samples})}}$$

smaller $\Delta t \to$ can see larger freqs

more samples = longer signal $\to$ ~~smaller~~ better res in freq.