The eigenvalues λ of a self-adjoint matrix $\boldsymbol{M} \in \mathbb{C}^{n\times n}$ are real, $\lambda \in \mathbb{R}$. A unitary operator has eigenvalues with unit modulus, $|\lambda| = 1$. The eigenvalues of an anti-Hermitian matrix are purely imaginary. For a normal matrix, the eigenvectors $\boldsymbol{v_1}, \boldsymbol{v_2} \in \mathbb{C}^n$ related to two different eigenvalues λ_1, $\lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$ are orthogonal, $\langle \boldsymbol{v_1}|\boldsymbol{v_2}\rangle = 0$. Scalar product: $\langle \boldsymbol{x}|\boldsymbol{y}\rangle = \sum_{i=1}^n x_i^* y_i$, $\boldsymbol{x} = (x_i), \boldsymbol{y} = (y_i) \in \mathbb{C}^n$.
A self-adjoint/unitary matrix is a particular case of a normal matrix. A self-adjoint matrix is positive definite if and only if all its eigenvalues are strictly positive.

Gram Schmidt Orthogonalization process:
Linearly independent vectors: $\{\boldsymbol{\chi_i}\}_{i=1,...;n}$
Orthogonal vectors: $\boldsymbol{\psi_i} = \boldsymbol{\chi_i} - \sum_{j=1}^{i-1}\langle \boldsymbol{\phi_j}|\boldsymbol{\chi_i}\rangle \boldsymbol{\phi_j}$
Orthonormal vectors: $\boldsymbol{\phi_i} = \boldsymbol{\psi_i}/\|\boldsymbol{\psi_i}\|$

Operators Definitions:
Self-adjoint/Hermitian: $\boldsymbol{H} = \boldsymbol{H}^\dagger \iff \langle \boldsymbol{\psi}|H\boldsymbol{\chi}\rangle = \langle H\boldsymbol{\psi}|\boldsymbol{\chi}\rangle$.
The composition of two self-adjoint matrices is itself self-adjoint if and only if the two matrices commute.
anti-Hermitian: $\boldsymbol{H} = -\boldsymbol{H}^\dagger$
Unitary: $\boldsymbol{U}^{-1} = \boldsymbol{U}^\dagger \iff \langle U\boldsymbol{\psi}|U\boldsymbol{\chi}\rangle = \langle \boldsymbol{\psi}|\boldsymbol{\chi}\rangle$.
Normal: $[\boldsymbol{A}, \boldsymbol{A}^\dagger] = \boldsymbol{A}\boldsymbol{A}^\dagger - \boldsymbol{A}^\dagger\boldsymbol{A} = 0$.

Spectral Theorem For a normal operator on an inner product space V over $\mathbb{C}$, there always exists an associated orthonormal eigenbasis of V. $\iff$
For a normal matrix $\boldsymbol{M} \in \mathbb{C}^{n\times n}$ there always exists a unitary matrix $\boldsymbol{U} \in \mathbb{C}^{n\times n}$ such that $\boldsymbol{D} = \boldsymbol{U}^\dagger \boldsymbol{M}\boldsymbol{U}$.

Existence of common orthonormal eigenbasis Two normal matrices $\boldsymbol{A}$, $\boldsymbol{B} \in \mathbb{C}^{n\times n}$ are simultaneously diagonalizable if and only if they commute ($[\boldsymbol{A}, \boldsymbol{B}] = 0$).

Eigenvalue problem $\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0 \implies \{\lambda_i\}$,
$(\boldsymbol{A} - \lambda_i\boldsymbol{I})\boldsymbol{v_i} = 0 \implies \{\boldsymbol{v_i}\}$.

Simultaneous diagonalisation for $\boldsymbol{A}$, $\boldsymbol{B} \in \mathbb{C}^{n\times n}$:
The eigenvector $\boldsymbol{v}$ spaning the one-dimensional eigenspace associated to $\boldsymbol{A}$ relative to the eigenvalue λ_A must be an eigenvector of $\boldsymbol{B}$ as well. Therefore, $\boldsymbol{v}$ must be colinear to one of the vectors of a common eigenbasis of $\boldsymbol{A}$ and $\boldsymbol{B}$. Same reasoning applies for the eigenvector $\boldsymbol{w}$ spaning the one-dimensional eigenspace associated to $\boldsymbol{B}$ relative to the eigenvalue λ_B. $\implies \{\boldsymbol{u_1} = \boldsymbol{v}/\|\boldsymbol{v}\|, \boldsymbol{u_2} = \boldsymbol{w}/\|\boldsymbol{w}\|, \boldsymbol{u_3} = \boldsymbol{u_1} \times \boldsymbol{u_2}\}$ forms a common orthonormal eigenbasis of $\boldsymbol{A}$ and $\boldsymbol{B}$.

Scalar 1st order ODE linear: $y' = a(x)y + b(x)$, $y(x_0) = y_0 \implies y(x) = y_0\, exp[\int_{x_0}^x a(s)ds] + \int_{x_0}^x b(s)\, exp[\int_s^x a(t)dt]ds$.
Existence and uniqueness of solution for IVP.

System of 1st order ODEs linear with constant matrix: $\frac{d\boldsymbol{y}(x)}{dx} = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{b}(x) \implies$
$\boldsymbol{y}(x) = exp[(x - x_0)\boldsymbol{A}]\boldsymbol{y_0} + \int_{x_0}^x exp[(x - s)\boldsymbol{A}]\boldsymbol{b}(s)ds$.
Existence and uniqueness of solution for IVP.
$\exp(\boldsymbol{A}) = \boldsymbol{P}\exp(\boldsymbol{D_A})\boldsymbol{P}^{-1}$. $\exp(x\boldsymbol{A}) = \boldsymbol{P}\exp(x\boldsymbol{D_A})\boldsymbol{P}^{-1}$
If $[\boldsymbol{A}, \boldsymbol{B}] = 0$, then $\exp(\boldsymbol{A} + \boldsymbol{B}) = \exp(\boldsymbol{A})\exp(\boldsymbol{B})$.

Scalar higher order ODE $\frac{d^n y(x)}{dx^n} = \phi(x, y, y', ..., y^{(n-1)})$
Linear: $\frac{d^n y(x)}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y(x)}{dx^{n-1}} + ... + a_0 y(x) = b(x)$.
Cast into a system of n 1st order ODE:
$\frac{d^n \boldsymbol{z}(x)}{dx^n} = \left(z_2, ..., z_n, \phi(x, y, y', ..., y^{(n-1)})\right)^T$
If the coefficients a_i are constants, solve the homogeneous equation with the *characteristic equation*:
$\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0 \implies$
If the roots are distinct: $y_i(x) = exp(\lambda_i x)$.
If a root has multiplicity $m > 1$: $y_j(x) = x^j exp(\lambda x)$, $j = 0, ..., m - 1$.

Wronskian Given a set of n solutions $\{y_i(x)\}_{i=1,...,n}$ to the homogeneous linear ODE of order n $\left(\frac{d^n y(x)}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y(x)}{dx^{n-1}} + ... + a_0 y(x) = 0\right)$.

$$W(y_1, ..., y_n)(x) = \begin{vmatrix} y_1(x) & \dots & y_n(x) \\ \vdots & \dots & \vdots \\ y_1^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix}$$

If $W(x) \neq 0$, the set of solutions $\{y_i(x)\}_{i=1,...,n}$ are *linearly independent*. If $W(x) = 0$, they are *linearly dependent*.
The wronskian verifies $\frac{dW}{dx} = -a_{n-1}(x)W(x)$.

Abel's identity: $W(x) = W(x_0)\, exp[-\int_{x_0}^x a_{n-1}(s)ds]$.

Particular solution: $y_p(x) = \sum_{i=1}^n y_i(x)\int^x \frac{W_i(s)}{W(s)}ds$,

$$\text{where } W_i(s) = \begin{vmatrix} y_1(s) & \dots & 0 & \dots & y_n(s) \\ \vdots & \dots & \vdots & \dots & \vdots \\ \vdots & \dots & 0 & \dots & \vdots \\ y_1^{(n-1)}(s) & \dots & \underbrace{b(s)}_{i^{\text{th}} column} & \dots & y_n^{(n-1)}(s) \end{vmatrix}.$$

Scalar linear 2nd order ODE $y'' + p(x)y' + q(x)y = r(x)$.
Solution: $y(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x)$, $C_1, C_2 \in \mathbb{R}$ Systematic procedure:
1) Obtain $y_1(x)$ with Frobenius Method or obvious solution.
2) Deduce $y_2(x)$ with Frobenius or the general relation:
$y_2(x) = y_1(x)\int^x \frac{ds}{y_1(s)^2} exp\left[-\int^s p(t)dt\right]$.
3) Find $y_p(x)$ with *variation of parameter* method: $y_p(x) = y_2(x)\int^x \frac{r(s)y_1(s)}{W(y_1,y_2)(s)}ds - y_1(x)\int^x \frac{r(s)y_2(s)}{W(y_1,y_2)(s)}ds$

Frobenius method write $y(x)$ in the form:
$y(x) = \sum_{j=0}^\infty a_j(x - x_0)^{j+s}$, $a_0 \neq 0$, $a_j \in \mathbb{C}$, $s \in \mathbb{C}$,
$y'(x) = \sum_{j=0}^\infty (j + s)a_j(x - x_0)^{j+s-1}$,
$y''(x) = \sum_{j=0}^\infty (j + s)(j + s - 1)a_j(x - x_0)^{j+s-2}$.
Plug into the homogeneous ODE and regroup by powers of $(x - x_0)$. All the coefficients must be 0. The coefficient of the lowest power gives the indicial equation. The a_j are determined using the reccurence relation provided by setting the coefficients of higher powers of $(x - x_0)$ to 0.

Fuch's theorem Around an *ordinary* point, Frobenius gives 2 linearly independent solutions (with $s_1 = 1, s_2 = 0$). Around an *regular singular* point, Frobenius gives at least 1 solution ($s \in \mathbb{C}$).
ordinary point: $p(x), q(x)$ are finite at $x = x_0$ and can be expanded as positive integer power series about x_0.
$p(x) = \sum_{j=0}^\infty p_j(x - x_0)^j$, $\quad q(x) = \sum_{j=0}^\infty q_j(x - x_0)^j$.
regular singular point: $p(x), q(x)$ can be expanded as
$p(x) = \sum_{j=-1}^\infty p_j(x - x_0)^j$, $\quad q(x) = \sum_{j=-2}^\infty q_j(x - x_0)^j$.

General solution $I(s) = s^2 + (p_{-1} - 1)s + q_{-2} = 0$.
1) $s_{1,2} = \alpha \pm i\beta$: $y_1(x) = \sum_{j=0}^\infty a_j(x - x_0)^{j+\alpha+i\beta}$, $y_2(x) = y_1^*(x)$
2) $s_1 = s_2 = s$: $y_1(x) = \sum_{j=0}^\infty a_j(x - x_0)^{j+s}$
$y_2(x) = y_1(x)\log(|x - x_0|) + \sum_{j=0}^\infty b_j(x - x_0)^{j+s}$
3) $s_1 > s_2 \in \mathbb{R}$: $y_1(x) = \sum_{j=0}^\infty a_j(x - x_0)^{j+s_1}$,
If $(s_1 - s_2) \notin \mathbb{N}$, $y_2(x) = \sum_{j=0}^\infty a_j(x - x_0)^{j+s_2}$.
Else, $y_2(x) = Cy_1(x)\log(|x - x_0|) + \sum_{j=0}^\infty b_j(x - x_0)^{j+s_2}$

BCP *Dirichlet*: Imposed values $y(a) = y_a, y(b) = y_b$.
Neumann: Imposed derivatives $y'(a) = y'_a, y'(b) = y'_b$.
Robin: Weighted combination of Dirichlet and Neumann.
Existence and uniqueness of the solution is not ensured.

Sturm-Liouville eigenvalue problem

$$\begin{cases} \mathcal{L}y = -\dfrac{1}{w(x)}\dfrac{d}{dx}\left[p_0(x)\dfrac{dy}{dx}\right] + q(x)y = \lambda y \\ \text{Homogeneous BCs such that } \mathcal{L} \text{ is self-adjoint w.r.t.:} \\ \langle f|g\rangle = \displaystyle\int_a^b w(x)f^*(x)g(x)dx, \quad p_0(x), q(x), w(x) \in \mathbb{R}, \\ \text{continuous and } p_0(x), w(x) > 0 for x \in]a,b[. \end{cases}$$

In order to prove self adjointness, show the relation $\langle f|\mathcal{L}g\rangle = \langle \mathcal{L}f|g\rangle$ by integrating by parts two times. Positive definiteness: $\langle f|\mathcal{L}f\rangle \geq 0$ and $\langle f|\mathcal{L}f\rangle = 0 \iff f(r) = 0$.

Casting an ODE into self-adjoint form Starting from: $a_2(x)y'' + a_1(x)y' + a_0(x)y = \lambda y$, one finds:

$$w(x) = \frac{C}{a_2(x)}\exp\left(\int^x \frac{a_1(s)}{a_2(s)}ds\right),$$
$$p_0(x) = -a_2(x)w(x), \qquad q(x) = a_0(x).$$

with C an arbitrary constant, except for its sign chosen to ensure that the weight function $w(x)$ is positive.
$x = \kappa r, \lambda = \kappa^2$ is a useful substitution to collapse a Sturm Liouville problem into a special ODE (Bessel, Laguerre, etc.).

Fourier series

Full series for an interval $[-L, L]$,
$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx\pi/L}, \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(s)e^{-ins\pi/L}ds$
$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{nx\pi}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{nx\pi}{L}\right)$
$a_n = \frac{1}{L}\int_{-L}^{L} f(s)\cos\left(\frac{ns\pi}{L}\right)ds, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(s)\sin\left(\frac{ns\pi}{L}\right)ds.$
For *even* function : $b_n = 0$, For *odd* function : $a_n = 0$.

Half-range series of aperiodic $f(x)$ over $[a, b]$:
Set $L = b - a$, $z = x - a$.
Cosine: Define $\tilde{f}(z) = f(z)$ if $z \in [0, L]$ and $\tilde{f}(z) = f(-z)$ if $z \in [-L, 0]$. Then, $a_n = \frac{2}{L}\int_0^L f(s)\cos\left(\frac{ns\pi}{L}\right)ds$ for $n \geq 0$
Sine: Define $\tilde{f}(z) = f(z)$ if $z \in [0, L]$ and $\tilde{f}(z) = -f(-z)$ if $z \in [-L, 0]$. Then $b_n = \frac{2}{L}\int_0^L f(s)\sin\left(\frac{ns\pi}{L}\right)ds$.

Half-range series of $f(x, y)$ over $[0, L_x] \times [0, L_y]$:
Sine: $f(x) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} c_{k,l} \sin\left(\frac{kx\pi}{L_x}\right)\sin\left(\frac{ly\pi}{L_y}\right)$.
$c_{k,l} = \frac{4}{L_x L_y}\int_0^{L_x}\int_0^{L_y} f(x, y)\sin\left(\frac{kx\pi}{L_x}\right)\sin\left(\frac{ly\pi}{L_y}\right)dxdy.$

Parseval's identity :
$\langle f(s)|g(s)\rangle = \int_{-L}^{L} g(s)f(s)^* ds = 2L\sum_{n=-\infty}^{\infty} g_n f_n^*.$

Parseval's theorem :
$\langle f(s)|f(s)\rangle = 2L\sum_{n=-\infty}^{\infty}|c_n|^2 = L\left(\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)\right)$

Convolution theorems :
$(f * g)(x) = \int_{-L}^{L} g(s)f(x - s)ds = 2L\sum_{n=-\infty}^{\infty} f_n g_n e^{in\pi x/L}.$
$f(x)g(x) = h(x) = \sum_{n=-\infty}^{\infty} h_n e^{in\pi x/L}$ with
$h_n = \sum_{m=-\infty}^{\infty} g_m f_{n-m}.$

Solving PDEs with BCs using Fourier series :
1) Obtain ODEs by *separation of variables* method.
2) Solve ODEs satisfying boundary conditions.
3) Solve PDE satisfying initial conditions:
Decompose initial conditions in half-range Fourier series.
For Dirichlet BC, choose sine series.
For Neumann BC, choose cosine series.

Delta Function :
$\delta(x) = 0 \quad \forall x \neq 0, x \in \mathbb{R}.\ f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx.$
$\delta_n(x) = \frac{n}{\sqrt{\pi}}e^{-n^2x^2}$ or $\delta_n(x) = \frac{\sin(nx)}{\pi x}$
$\delta_n(q - k) = \frac{1}{2\pi}\int_{-n}^{n} e^{is(q-k)}ds$
$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\delta_n(x)dx = \int_{-\infty}^{\infty} f(x)\delta(x)dx$

Fourier transform

$\mathcal{F}[f(t)] = g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s)e^{i\omega s}ds.$
$\mathcal{F}^{-1}[g(\omega)] = f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{-i\omega t}d\omega.$
Of a *Gaussian* function ($Re(a) \geq 0$) :
$\mathcal{F}[f(t)] = g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-as^2}e^{i\omega s}ds = \frac{1}{\sqrt{2a}}\exp\left(-\frac{\omega^2}{4a}\right).$
Of *unity* :
$\mathcal{F}[1] = g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega s}ds = \lim_{n\to\infty}\sqrt{2\pi}\delta_n(\omega).$
Of *delta function* :
$\mathcal{F}[\delta(t - a)] = g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(t - a)e^{i\omega s}ds = \frac{1}{\sqrt{2\pi}}e^{i\omega a}.$
In 3D space :
$\mathcal{F}[f(\vec{r})] = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} f(\vec{s})e^{i\vec{k}\cdot\vec{s}}d^3s.$
$\mathcal{F}^{-1}[g(\vec{k})] = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} g(\vec{k})e^{-i\vec{k}\cdot\vec{r}}d^3k.$

Elementary properties : given $\mathcal{F}[f(\vec{r})] = g(\vec{k})$,
$\mathcal{F}[f(\vec{r} - \vec{R})] = e^{i\vec{R}\cdot\vec{k}}g(\vec{k})$. $\mathcal{F}[f(\alpha\vec{r})] = \frac{1}{|\alpha|^3}g(\alpha^{-1}\vec{k}), \alpha > 0.$
$\mathcal{F}[f(-\vec{r})] = g(-\vec{k}). \qquad \mathcal{F}[f^*(-\vec{r})] = g^*(\vec{k}).$
$\mathcal{F}[\vec{\nabla}f(\vec{r})] = -i\vec{k}g(\vec{k}). \qquad \mathcal{F}[\nabla^2 f(\vec{r})] = -\vec{k}^2 g(\vec{k}).$
$\mathcal{F}\left[\int_{-\infty}^{x} f(s)ds\right] = -\frac{g(\omega)}{i\omega} + \pi g(0)\delta(\omega).$

Fourier convolution :
$(V * U)(r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(s)V(r - s)ds.$
$(V * U)(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} U(\vec{s})V(\vec{r} - \vec{s})d^3s.$

1st convolution theorem Defining $\hat{f}(\vec{k}) = \mathcal{F}[f(\vec{r})]$,
$\int_{-\infty}^{\infty} U(s)V(r - s)ds = \int_{-\infty}^{\infty}\hat{U}(k)\hat{V}(k)e^{-ikr}dk.$
$\int_{\mathbb{R}^3} U(\vec{s})V(\vec{r} - \vec{s})d^3s = \int_{\mathbb{R}^3}\hat{U}(\vec{k})\hat{V}(\vec{k})e^{-i\vec{k}\cdot\vec{r}}d^3k.$
This result can be proven by replacing the definition of the inverse FT of U and V on the RHS, using Fubini's theorem and the definition of the delta sequence $\delta_n(q - k)$.

Parseval's theorem :
$\int_{-\infty}^{\infty} U(s)W^*(s)ds = \int_{-\infty}^{\infty}\hat{U}(k)\hat{W}^*(k)dk$
$\iff \langle U|W\rangle = \langle\hat{U}|\hat{W}\rangle$. One therefore has : $\|U\|^2 = \|\hat{U}\|^2$.

Solving unbounded PDEs using Fourier transforms
1) Apply Fourier transform to both sides (with elementary properties).
2) Solve the transformed equation using transformed initial conditions: $\hat{\psi}_0(k) = \hat{\psi}(t = 0), \quad \hat{\nu}_0(k) = \frac{\partial\hat{\psi}}{\partial t}\Big|_{t=0}$.
3) Apply inverse Fourier transform to obtain the solution.
4) Compare with the convolution theorem form. If the PDE is of higher dimension, first apply $\mathcal{F}_y$, then $\mathcal{F}_x$ (or inversely). When applying $\mathcal{F}_x$, keep in mind the following relations: $\mathcal{F}_x(\frac{\partial^2\psi}{\partial y^2}) = \frac{\partial^2\mathcal{F}_x(\psi)}{\partial y^2}$ and $\mathcal{F}_x(\frac{\partial^2\psi}{\partial x^2}) = -k_x^2\mathcal{F}_x(\psi)$

Useful trigonometric relations :
$\sin(x) + \sin(y) = 2\sin(\frac{x+y}{2})\cos(\frac{x-y}{2}).$
$\sin(x)\sin(y) = \frac{\cos(x-y) - \cos(x+y)}{2}.$
$\cos(x) - \cos(y) = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})$