Chap. 4

Fourier Series and Applications to PDEs

Reference: Chapters 19 & 9 Arfken

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Content of this chapter

4. Fourier Series and Applications to PDEs
   - Representation of periodic functions, including those with discontinuities
   - Half range series expansion representation and aperiodic functions
   - Operations on Fourier series and convergence
   - Power spectrum and Parseval's identity
   - Summation of Fourier Series (tables or brute force)
   - Applications to homogeneous linear PDE's
   - Types of PDE's, and boundary conditions
   - Solving multidimensional problems with multiple orthogonal basis
   - Double Fourier series for wave equation over a rectangular domain
   - Using Fourier-Bessel series for wave equation in circular geometry
   - Using Fourier series to solve heat equation with Neumann B. C.

Introduction to Fourier Series

- **Fourier Series** are a basic tool allowing us to represent any wave-like function as a superposition of sine (and cosine) waves. The common name for the field is **Fourier Analysis**.
- As well as being useful for spectral analysis, Fourier series assist solving some types of ODE's, and PDE's, as we shall see.
- The basic idea is that any piecewise regular function (only a finite number of finite discontinuities, or finite number of extremas) in a specified interval $[-\pi, \pi]$ can be represented by a sum of sine and cosine functions, or equivalently complex exponentials.
- Fourier Series can be derived from the mathematical framework developed in the Hilbert Space chapter of this course.

Fourier Series as an Orthogonal Expansion of a Vector

We have seen at the start of the "Hilbert Space" chapter that given an orthogonal basis $\{\boldsymbol{\phi}_n\}_{n=M,\dots,N}$ of a Hilbert space V over $\mathbb{C}$, the components $x_n \in \mathbb{C}$ of an *abstract* vector $\boldsymbol{\psi}$ on this basis are obtained by **projection**:

$$\boldsymbol{\psi} = \sum_{n=M}^{N} x_n \boldsymbol{\phi}_n \qquad \Longrightarrow \qquad \langle \boldsymbol{\phi}_n | \boldsymbol{\psi} \rangle = \langle \boldsymbol{\phi}_n | \sum_{j=M}^{N} x_j \boldsymbol{\phi}_j \rangle = \sum_{j=M}^{N} x_j \underbrace{\langle \boldsymbol{\phi}_n | \boldsymbol{\phi}_j \rangle}_{=0,\text{if} n \neq j} = x_n \langle \boldsymbol{\phi}_n | \boldsymbol{\phi}_n \rangle$$

$$\Longrightarrow \qquad \boxed{x_n = \frac{\langle \boldsymbol{\phi}_n | \boldsymbol{\psi} \rangle}{\langle \boldsymbol{\phi}_n | \boldsymbol{\phi}_n \rangle}}.$$

We define the Hilbert space $L^2_w([a,b])$, with weight function $w = 1$ over $[a,b] = [-\pi, \pi]$, $< f|g >= \int_a^b ds\, f^* g$. The orthogonal basis is chosen such that $\boldsymbol{\phi}_n = \exp(inx)$, and we allow $M \to -\infty$ and $N \to \infty$. Letting $c_n = x_n$, and removing the abstract vector notation (in this case $\boldsymbol{\psi}$ is replaced by $f(x)$):

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{4.1}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} ds\, f(s) e^{-ins} \tag{4.2}$$

The basis $\boldsymbol{\phi}_n = \exp(inx)$ is indeed orthogonal, but the basis is not orthonormal:

$$\langle \boldsymbol{\phi}_n | \boldsymbol{\phi}_m \rangle \equiv \int_a^b dx\, \boldsymbol{\phi}_n^* \boldsymbol{\phi}_m = \int_{-\pi}^{\pi} ds\, e^{-ins} e^{+ims} = 2\pi \delta_{n,m}$$

Sturm Liouville Theory and Convergence in the Mean

The ODE

$$\mathcal{D}\phi(x) = -\frac{d^2}{dx^2}\phi(x) = \lambda\phi(x),$$

on $[-\pi, \pi]$, and $x \in \mathbb{R}$, with e.g. periodic boundary conditions $\phi(-\pi) = \phi(\pi)$, $\phi'(-\pi) = \phi'(\pi)$ is a **Sturm-Liouville** problem. These boundary conditions ensure that the operator $\mathcal{D}$ is self adjoint with respect to the product $< f|g >$.

As a consequence, the system has eigenfunctions $\exp(inx)$ which form a complete set, with eigenfunctions of different eigenvalues n^2 orthogonal (n integer), i.e.

$$\left\langle e^{inx} | e^{imx} \right\rangle = 2\pi\delta_{m,n}$$

An orthonormal set of eigenfunctions can easily be established, and these functions are $\boldsymbol{\phi}_n / \sqrt{\langle \boldsymbol{\phi}_n | \boldsymbol{\phi}_n \rangle} = \exp(inx)/\sqrt{2\pi}$.

Since the eigenfunctions of a Sturm-Louiville problem forms a complete orthogonal basis, the Fourier expansions of L^2 functions will at least converge in the mean.

Definition of series $f(x)$ and Convergence in the Mean

An L^2 function $f(x)$ has the property that the integral exists:

$$\int_{-\pi}^{\pi} dx |f(x)|^2 < \infty.$$

Defining the following partial sum

$$f_N(x) = \sum_{n=-N}^{N} c_n \, \mathrm{e}^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} ds \, f(s) \mathrm{e}^{-ins},$$

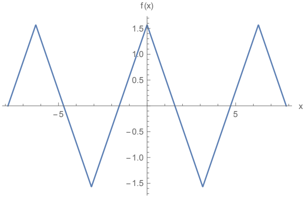
convergence to the mean of the L^2 function $f(x)$ requires that

$$\boxed{\lim_{N\to\infty} ||f(x) - f_N(x)|| \equiv \lim_{N\to\infty} \left[\int_{-\pi}^{\pi} dx |f(x) - f_N(x)|^2 \right]^{1/2} = 0} \tag{4.3}$$

Example: representation of a continuous periodic function

Let us take a Triangular wave, with period 2π, defined over $[-\pi, \pi]$ as:

$$f(x) = \begin{cases} x + \pi/2 & \text{for } -\pi \leq x \leq 0 \\ -x + \pi/2 & \text{for } 0 \leq x \leq \pi, \end{cases}$$



The Fourier Series is,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

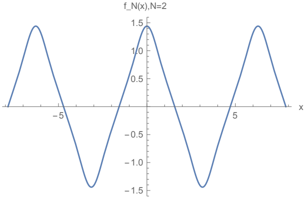
with

$$\begin{aligned} c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} ds\, e^{-ins} f(s) &= \frac{1}{2\pi}\int_{-\pi}^{0} ds\, e^{-ins}\left(s + \frac{\pi}{2}\right) + \frac{1}{2\pi}\int_{0}^{\pi} ds\, e^{-ins}\left(-s + \frac{\pi}{2}\right) \\ c_n &= \frac{1}{\pi n^2}\left(1 - (-1)^{|n|}\right). \end{aligned}$$

Representation of a continuous periodic function

The series representation of this triangular wave is this:

$$\begin{aligned} f(x) &= \frac{2}{\pi}\left(+\ldots+\frac{\exp(-i3x)}{3^2}+\frac{\exp(-ix)}{1^2}+\frac{\exp(ix)}{1^2}+\frac{\exp(i3x)}{3^2}+\ldots+\right) \\ &= \frac{1}{\pi}\sum_{n=-\infty}^{\infty}(1-(-1)^{|n|})\frac{\exp(inx)}{n^2} \\ &= \frac{1}{\pi}\sum_{n=-\infty}^{\infty}(1-(-1)^{|n|})\frac{\cos(nx)}{n^2} = \frac{2}{\pi}\sum_{n=1}^{\infty}(1-(-1)^{|n|})\frac{\cos(nx)}{n^2} \\ &= \frac{4}{\pi}\sum_{p=1}^{\infty}\frac{\cos[(2p-1)x]}{(2p-1)^2}. \end{aligned}$$



The figure f_N(x) with $N = 2$ is a truncation of the infinite series, with $2N = 4$ terms. **Could we have chosen an alternative trigonometric polynomial of degree 4 with smaller error?**

Partial Summation of Fourier Series and best approximation over interval

- Convergence to the mean of an L^2 function $f(x)$ requires that

$$\lim_{N\to\infty} ||f(x) - f_N(x)||^2 = \lim_{N\to\infty} \int_{-\pi}^{\pi} dx |f(x) - f_N(x)|^2 = 0$$

with

$$f_N(x) = \sum_{n=-N}^{N} c_n \, \mathrm{e}^{inx}, \quad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} ds \, f(s) \mathrm{e}^{-ins}$$

- In practice $f(x)$ is often represented by a finite number of coefficients. We now ask if $f_N(x)$ provides the smallest error, or whether another trigonometric polynomial

$$\mathcal{F}_N(x) = \sum_{n=-N}^{N} \mathcal{C}_n \mathrm{e}^{inx},$$

with other coefficients C_n, could produce a smaller error. We aim to minimise the global error

$$E[\mathcal{F}_N] = ||f(x) - \mathcal{F}_N(x)||^2$$

Partial Summation of Fourier Series and best approximation over interval

Exploiting the orthogonal properties of the expansion, a straightforward exercise shows that

$$E[\mathcal{F}_N] = \langle f|f\rangle - 2\pi \sum_{n=-N}^{N} \left(c_n^\star \mathcal{C}_n + c_n \mathcal{C}_n^\star - |\mathcal{C}_n|^2\right).$$

This can be compared with the specific case $E[f_N]$ where we take $\mathcal{F}_N$ to be f_N, i.e. f truncated to N terms, so that we set $\mathcal{C}_k = c_k$, giving:

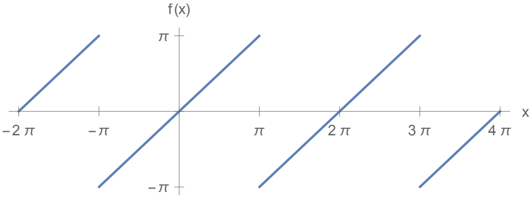
$$E[f_N] = \langle f|f\rangle - 2\pi \sum_{n=-N}^{N} |c_n|^2.$$

The difference in error is

$$E[\mathcal{F}_N] - E[f_N] = 2\pi \sum_{n=-N}^{N} |\mathcal{C}_n - c_n|^2$$

Thus, $E[f_N] \leq E[\mathcal{F}_N]$. Equality iff $\mathcal{C}_k = c_k$ and $\mathcal{F}_k = f_k$. This agrees with the Hilbert Space section for orthogonal series: **an Nth order trigonometric polynomial most accurately represents $f(x)$ if it is a full Fourier series truncated at the Nth coefficient.**

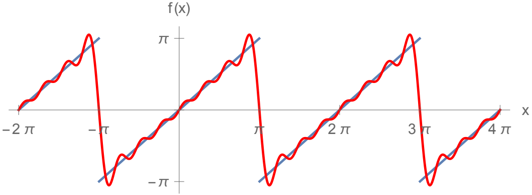
Representation of Discontinuous functions



- Consider the sawtooth wave above with period 2π. It is piecewise continuous, but has discontinuity at $x = x_0 = \pi$. Could this function be represented by a Taylor series around some point with an infinite convergence radius? No, such a power series will converge only within a radius of convergence up to the nearest singularity.
- But, **a Fourier series uses information from the entire expansion interval**. Moreover, the function will converge in the mean. For a function $f(x)$ **that is discontinuous at point x_0, it's Fourier series is the arithmetic average of the left and right approaches:**

$$\lim_{N\to\infty} f_N(x_0) = \lim_{\epsilon\to 0} \left[\frac{f(x_0+\epsilon) + f(x_0-\epsilon)}{2} \right]$$

Representation of Discontinuous functions



The Fourier series representation of a sawtooth wave is represented by the following in the limit $N \to \infty$:

$$f_N(x) = 2 \sum_{n=1}^{N} (-1)^{n+1} \frac{\sin nx}{n}$$

The sawtooth wave is discontinuous at point $x_0 = \pi$, but its Fourier series is indeed the arithmetic average of the left and right approaches:

$$\begin{aligned} \lim_{N \to \infty} f_N(x = \pi) &= 0 \\ &= \lim_{\epsilon \to 0} \left[\frac{f(\pi + \epsilon) + f(\pi - \epsilon)}{2} \right] \end{aligned}$$

2π periodic Fourier Series

The complex exponential Fourier series of Eqs. (4.1) and (4.2) is identical to:

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx, && (4.4) \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} ds\, f(s) \cos ns \quad n = 0, 1, 2, \ldots, && (4.5) \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} ds\, f(s) \sin ns \quad n = 0, 1, 2, \ldots, && (4.6)
\end{aligned}
$$

The relations to the complex exponential coefficients are:

$$
c_n = \frac{1}{2}(a_n - ib_n) \text{ and } c_{-n} = \frac{1}{2}(a_n + ib_n), \quad n \geq 0. \tag{4.7}
$$

Fourier cosine series: For even functions $f(-x) = f(x)$, $\Rightarrow b_n = 0$:

$$
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.
$$

Fourier sine series: For odd functions $f(-x) = -f(x)$, $\Rightarrow a_n = 0$:

$$
f(x) = \sum_{n=1}^{\infty} b_n \sin nx.
$$

Notes

A Fourier series defined in terms of sines and cosines is easily derived from the exponential series:

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty}^{\infty} c_n \cos nx + \sum_{n=-\infty}^{\infty} i c_n \sin nx \\
&= \sum_{n=-\infty}^{-1} c_n \cos nx + c_0 + \sum_{n=1}^{\infty} c_n \cos nx + \sum_{n=-\infty}^{-1} i c_n \sin nx + \sum_{n=1}^{\infty} i c_n \sin nx \\
&= \sum_{n=1}^{\infty} (c_n + c_{-n}) \cos nx + c_0 + \sum_{n=1}^{\infty} i(c_n - c_{-n}) \sin nx \\
&= \sum_{n=1}^{\infty} a_n \cos nx + \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin nx
\end{aligned}
$$

with $a_0 = 2c_0, \;\; a_n = c_n + c_{-n}, \;\; b_n = i(c_n - c_{-n})$ where $n > 0$.

$$
\begin{aligned}
a_0 &= \frac{2}{2\pi} \int_{-\pi}^{\pi} ds\, e^{i0s} f(s) = \frac{1}{\pi} \int_{-\pi}^{\pi} ds\, f(s) \\
a_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} ds\, (e^{-ins} + e^{ins}) f(s) = \frac{1}{\pi} \int_{-\pi}^{\pi} ds\, f(s) \cos ns \\
b_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} ds\, i(e^{-ins} - e^{ins}) f(s) = \frac{1}{\pi} \int_{-\pi}^{\pi} ds\, f(s) \sin ns
\end{aligned}
$$

Half range sine and cosine Fourier series

- For a function $f(x)$ defined over the interval $[0, \pi]$, we can represent $f(x)$ as a **half range Fourier cosine series** by extending $f(x)$ over $[-\pi, \pi]$ and forcing it to be even $f(-x) = f(x)$:

$$a_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} ds\, f(-s)\cos ns + \int_{0}^{\pi} ds\, f(s)\cos ns\right) = \frac{2}{\pi}\int_{0}^{\pi} ds\, f(s)\cos ns$$

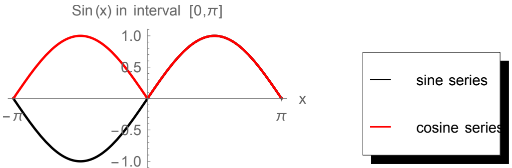
- We can represent the same function as a **half range Fourier sine series** by extending $f(x)$ over $[-\pi, \pi]$ and forcing it to be odd $f(-x) = -f(x)$:

$$b_n = \frac{1}{\pi}\left(-\int_{-\pi}^{0} ds\, f(-s)\sin ns + \int_{0}^{\pi} ds\, f(s)\sin ns\right) = \frac{2}{\pi}\int_{0}^{\pi} ds\, f(s)\sin ns$$

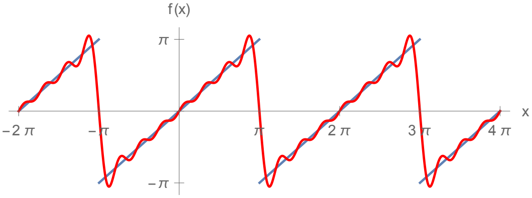
Take the case of $f(x) = \sin x$ over interval $[0, \pi]$:

$$\text{Fourier cosine series :}\quad f(x) \equiv \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad = \quad \frac{2}{\pi} + \sum_{p=1}^{\infty} \frac{4\cos 2px}{\pi\left[1-(2p)^2\right]}$$

$$\text{Fourier sine series :}\quad f(x) \equiv \sum_{n=1}^{\infty} b_n \sin nx \quad = \quad \sin(x)$$



Convergence consequences with piecewise discontinuity

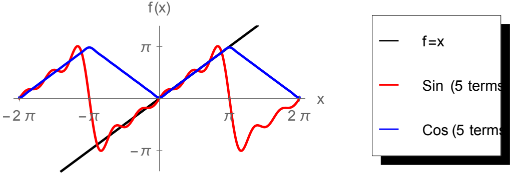


The example of a Fourier representation of the sawtooth had to be constructed by a **Fourier sine series** because the sawtooth function was defined with period 2π, and because the sawtooth function is **purely odd**. Using Eqs. (4.4) and (4.6) one can show (an exercise) that

$$f(x) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n} \tag{4.8}$$

- The discontinuity in $f(x)$ leads to the nth coefficient decreasing as $\mathcal{O}(1/n)$. Convergence of the series is conditional only.

Half range representation of aperiodic functions



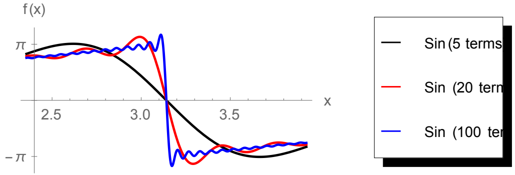
- Over $[0, \pi]$ the previous sine series represents $f(x) = x$.
- Within $[0, \pi]$ we are free to use either a **half range cosine series** or a **half range sine series** (or a combination) to represent an **aperiodic** function. e.g with the cosine series:

$$f(x) = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{4}{\pi} \frac{\cos(2n+1)x}{(2n+1)^2}$$

- The coefficients decrease as $\mathcal{O}(1/n^2)$, so that the series converges absolutely. This is true in general if there is not a discontinuity in $f(x)$, even if there are discontinuities in its derivatives.

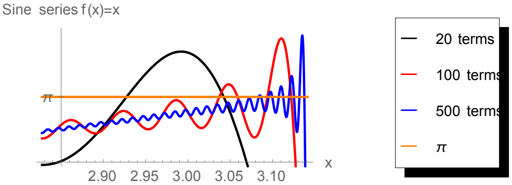
Half range representation choice important e.g. for ODE's and PDE's.

Local Accuracy of Fourier Series: Gibbs Phenomena



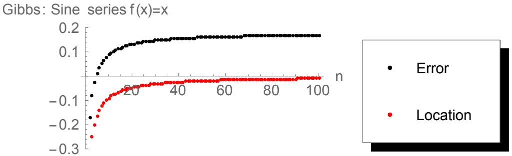
- Consider the Fourier representation of the sawtooth wave over the full period. Since it is odd, a sine series is required, and we know it is of the form of Eq. (4.8).
- The piecewise discontinuity causes slow convergence in the mean as more coefficients are added. But, the magnitude of the **local error** close to the discontinuity does not reduce with more terms. The position of the maximum overshoot simply gets closer to the discontinuity.
- The **Gibb's phenomenon** indicates that a Fourier representation may be highly unreliable for precise numerical work, especially near a discontinuity.
- The phenomenon is not limited to Fourier series, it occurs for other eigenfunctions too. Proof is reserved to additional reading. For interested readers see e.g. section 19.3 in Arfken.

Local Accuracy of Fourier Series: Gibbs Phenomena



The maximum in $f(x)$ occurs at $x = x_m$. As seen the maximum is larger than the correct value of the sawtooth amplitude π near the singularity at $x = x_0$. In the plots,

$$\text{Location} = \frac{x_m - x_0}{x_0} \quad \text{and} \quad \text{Error} = \frac{f(x_m) - \pi}{\pi}$$



Fourier expansion of functions with period $2L$

- Up until this point, the interval for establishing the Fourier series has been assumed $[-\pi, \pi]$. The resulting Fourier series are periodic over this interval. However, it is convenient to be able to set the interval to $[-L, L]$, especially if the function that the series represents has wavelength $2L$.
- For an interval $[-L, L]$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\pi x/L}. \tag{4.9}$$

Or equivalently

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{4.10}$$

The coefficients c_n, a_0, a_n and b_n are related to $f(x)$ through the following definite integrals (subject to the requirement that they exist):

$$c_n = \frac{1}{2L}\int_{-L}^{L} ds\, f(s)\mathrm{e}^{-in\pi s/L} \quad n = \ldots, -2, -1, 0, 1, 2, \ldots, \tag{4.11}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} ds\, f(s)\cos\frac{n\pi s}{L}, \quad n = 0, 1, 2, \ldots, \tag{4.12}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} ds\, f(s)\sin\frac{n\pi s}{L}, \quad n = 1, 2, \ldots, \tag{4.13}$$

Notes

For representing an aperiodic function $f(x)$ defined over interval $[0, L]$ using half range series

To conveniently construct a cosine series representation of an aperiodic function $f(x)$ defined over an arbitrary interval $[0, L]$, we construct an even function $\overline{f}(x)$ which is defined as $\overline{f}(x) = f(x)$ over interval $[0, L]$, and $\overline{f}(x) = f(-x)$ over $[-L, 0]$. For the series representation of $\overline{f}(x)$ one then has $b_n = 0$, while a_n will be obtained via Eq. (4.12). Clearly, integration is required over only **half the range** of $[-L, L]$, so that over $[0, L]$:

$$f(x) = \overline{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad \text{with} \quad a_n = \frac{2}{L}\int_0^L ds\, f(s) \cos\frac{n\pi s}{L}.$$

To conveniently construct a sine series representation of an aperiodic function $f(x)$ defined over an arbitrary interval $[0, L]$, we construct an odd function $\overline{f}(x)$ which is defined as $\overline{f}(x) = f(x)$ over interval $[0, L]$, and $\overline{f}(x) = -f(-x)$ over $[-L, 0]$. For the series representation of $\overline{f}(x)$ one then has $a_n = 0$, while b_n will be obtained via Eq. (4.13). Clearly, integration is required over only **half the range** of $[-L, L]$, so that over $[0, L]$:

$$f(x) = \overline{f}(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{with} \quad b_n = \frac{2}{L}\int_0^L ds\, f(s) \sin\frac{n\pi s}{L}.$$

Notes

For representing an aperiodic function $f(x)$ defined over arbitrary interval $[a, b]$ using half range series

To conveniently construct a cosine series representation of an aperiodic function $f(x)$ defined over an interval $[a, b]$, we first define the interval $[0, L]$, where $L = b - a$. We construct an even function $\overline{f}(z)$ which is defined as $\overline{f}(z) = f(z)$ over $[0, L]$, and $\overline{f}(z) = f(-z)$ over $[-L, 0]$, where $z = x - a$. For the series representation of $\overline{f}(z)$ one then has $b_n = 0$, while a_n will be obtained via Eq. (4.12). Clearly, integration is required over only **half the range** of $[-L, L]$, so that over $[0, L]$:

$$f(z) = \overline{f}(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi z}{L} \quad \text{with} \quad a_n = \frac{2}{L}\int_0^L ds\, f(s) \cos\frac{n\pi s}{L}$$

Translation of resulting expansion back to x uses simply $z = x - a$.

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$$f(z) = \overline{f}(z) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi z}{L} \quad \text{with} \quad b_n = \frac{2}{L}\int_0^L ds\, f(s) \sin\frac{n\pi s}{L}$$

Translation of resulting expansion back to x uses simply $z = x - a$.

Operations on Fourier Series and Convergence

Integration and differentiation operations on a Fourier series are particularly important for PDE and ODE problems.

Integration of a Fourier series Eq. (4.10) yields improved convergence, since the operation places n on the denominator of all the sinusoidal terms:

$$\int_{x_0}^{x} dx\, f(x) = \frac{a_0 L}{2\pi}(x - x_0) + \sum_{n=1}^{\infty} \frac{a_n L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)\Bigg|_{x_0}^{x} - \sum_{n=1}^{\infty} \frac{b_n L}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\Bigg|_{x_0}^{x}$$

The Fourier series of the integral of $f(x)$ may be convergent even when the series of $f(x)$ itself is not convergent.

Due to the secular contribution the actual Fourier series associated with the integral above is in fact

$$\int_{x_0}^{x} dx\, f(x) - \frac{a_0 L}{2\pi} x$$

Operations on Fourier Series and Convergence

- Term by term **differentiation of a Fourier series** is only possible if the series is **uniformally convergent**.
- For example, differentiation of the sine series representation of $f(x) = x$ (Eq. (4.8)), is not convergent. While $f(x)' = 1$, term by term differentiation of the Fourier sin series of $f(x) = x$ yields

$$\frac{d}{dx}[\text{sine series of } f(x) = x] = 2\sum_{n=1}^{\infty}(-1)^{n+1}\cos nx.$$

  which is not convergent (we should have chosen cosine representation!).
- In general, for the Fourier series $f(x)$ given by Eqs. (4.10) - (4.12), we have the following

$$\frac{d}{dx}f(x) = -\sum_{n=1}^{\infty}\frac{n\pi a_n}{L}\sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty}\frac{n\pi b_n}{L}\cos\left(\frac{n\pi x}{L}\right).$$

  However, as seen for the example above, one must be cautious when differentiating Fourier series.

Useful Elementary Properties of Fourier Series

Fourier Series is Linear. Consider $h(x) = \alpha f(x) + \beta g(x)$, with $f(x)$ and $g(x)$ given by series

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{in\pi x/L} \quad \text{and} \quad g(x) = \sum_{n=-\infty}^{\infty} g_n e^{in\pi x/L},$$

with Eq. (4.11) defining f_n or g_n in terms of f or g. It is then straightforward to obtain the linear relation:

$$h(x) = \sum_{n=-\infty}^{\infty} h_n e^{in\pi x/L} \quad \text{with} \quad h_n = \alpha f_n + \beta g_n$$

1st Convolution Theorem

$$(f * g)(x) \equiv \int_{-L}^{L} ds\, g(s) f(x-s) = 2L \sum_{n=-\infty}^{\infty} f_n g_n e^{in\pi x/L} \tag{4.14}$$

2nd Convolution Theorem. It can be shown that the product $h(x) = f(x)g(x)$ is given by a convolution of the coefficients c_n and d_n:

$$h(x) = \sum_{n=-\infty}^{\infty} h_n e^{in\pi x/L} \quad \text{with} \quad h_n = \sum_{m=-\infty}^{\infty} g_m f_{n-m}$$

Useful Elementary Properties of Fourier Series

Property for a Real function $f(x)$. For such a series $f(x) = \sum_{n=-\infty}^{\infty} f_n \mathrm{e}^{in\pi x/L}$ with the property,

$$f_n^* = \frac{1}{2L}\int_{-L}^{L} ds\, f(s)\mathrm{e}^{in\pi s/L} = f_{-n}$$

Translation Property. With $f(x) = \sum_{n=-\infty}^{\infty} f_n \mathrm{e}^{in\pi x/L}$, one can translate as follows:

$$\begin{aligned} f(x - x_0) &= \sum_{n=-\infty}^{\infty} f_n \mathrm{e}^{in\pi(x-x_0)/L} \\ &= \sum_{n=-\infty}^{\infty} \tilde{f}_n \mathrm{e}^{in\pi x/L}, \quad \tilde{f}_n = f_n \mathrm{e}^{-in\pi x_0/L} \end{aligned}$$

Parseval's Theorem and Power spectra

From the Hilbert space section of this course, we know that for two orthogonal basis expansions

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \mathrm{e}^{in\pi x/L} \quad \text{and} \quad g(x) = \sum_{n=-\infty}^{\infty} g_n \mathrm{e}^{in\pi x/L}$$

with

$$f_n = \frac{1}{2L}\int_{-L}^{L} ds\, f(s)\mathrm{e}^{-in\pi s/L} \quad \text{and} \quad g_n = \frac{1}{2L}\int_{-L}^{L} ds\, g(s)\mathrm{e}^{-in\pi s/L}$$

the scalar product of these series is (due to orthogonality) yields **Parseval's identity**:

$$\boxed{\langle f(s)|g(s)\rangle \equiv \int_{-L}^{L} ds\, g(s) f(s)^* = 2L \sum_{n=-\infty}^{\infty} g_n f_n^*.}$$

The result is a special case of the first convolution theorem (in Eq. (4.14), replace $f(x-s)$ with $f(s)^*$). Parseval's identity can easily be written in terms of the coefficients of equivalent sine/cosine expansions.

Parseval's Theorem and Power spectra

A series exercise will show the special case $g(x) = f(x)$, where it is assumed that the series $f(x)$ has uniform convergence. Using c_n in place of f_n and g_n (for contact with earlier notation, i.e. Eq. (4.11)) the result is:

$$\boxed{\langle f(s)|f(s)\rangle = 2L \sum_{n=-\infty}^{\infty} |c_n|^2 \equiv L\left[\frac{1}{2}|a_0|^2 + \sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right)\right]} \tag{4.15}$$

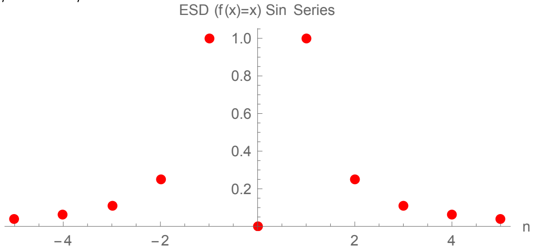
It can be further shown that this relation holds for any L^2 function, but the proof is beyond the scope of this course.

The so called **Energy Spectral Density (ESD)** is a decomposition of energy into frequencies, and is proportional to $|c_n|^2$.

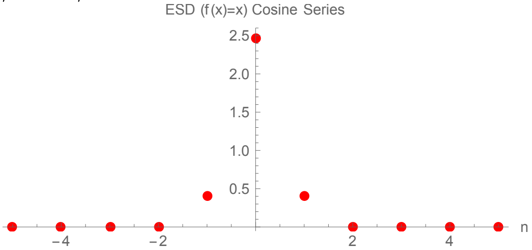
The total energy associated with $f(x)$ is proportional to $\langle f(x)|f(x)\rangle$. It is a sum over the ESD, i.e. a sum of $|c_n|^2$ over all n.

$|c_n|^2$ Energy Spectral Density (ESD) Examples

Sine series representation of $f(x) = x$ over $[0, \pi]$.
$|c_n|^2 = |b_n|^2/4 \sim 1/n^2$

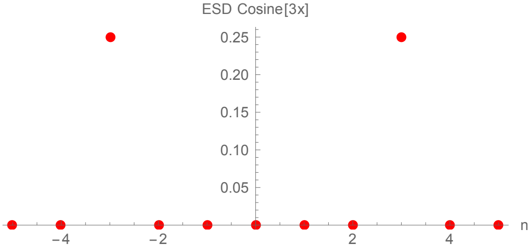


Cosine series representation of $f(x) = x$ over $[0, \pi]$.
$|c_n|^2 = |a_n|^2/4 \sim 1/n^4$

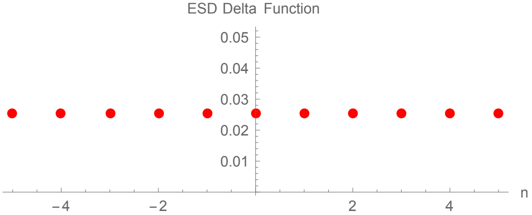


Spectrum width and $f(x)$ localisation

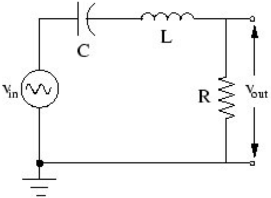
Series representation of $f(x) = \cos 3x$. $|c_n|^2$ only non-zero for $n = 3$.



Series representation of $f(x) = \delta(x)$. $|c_n|^2 \sim 1/n^0$. Infinite energy because $f(x)$ is not piecewise continuous, $\int ds\, \delta(s)^2 = \infty$.



ESD in Forced Oscillations - 2nd order non-homogeneous ODE



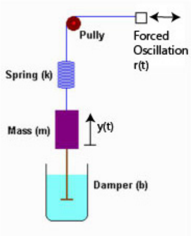
EMF conservation $V_L(t) + V_R(t) + V_C(t) = V_{in}(t)$ where $V_{in}(t)$ is a given voltage waveform from an oscillator and

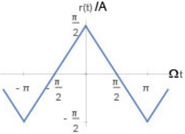
$$V_C(t) = \frac{1}{C}\int_{-\infty}^{t} I(t)dt \ , \ V_L(t) = L\frac{dI(t)}{dt} \text{ and } V_R(t) = RI(t)$$

where $I(t)$ is the current. Evidently we wish to measure the voltage across the resistor $V_{out}(t) = V_R = I(t)R$. Differentiating the EMF conservation equation, the system of equations is identified in the recognisable form of a second order non-homogeneous ODE for $I(t)$:

$$L\frac{d^2I(t)}{dt^2} + R\frac{dI(t)}{dt} + \frac{1}{C}I(t) = \frac{dV_{in}(t)}{dt}.$$

ESD in Forced Oscillations





The driven LRC problem is equivalent to solving for the height $y(t)$ of a mass m hanging from a damped spring, and driven with external force $r(t)$:

$$m\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = r(t)$$

The natural frequency of the homogeneous system is $\omega_0 = \sqrt{k/m}$. We take $r(t)$ to be the triangular waveform above. Note that for the RLC problem, with dV_{in}/dt triangular, then $V_{in}(t)$ is a square wave.

ESD in Forced Oscillations

We have already seen that the triangular wave form can be represented by

$$r(t) = A\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\Omega t}{(2n-1)^2} = A\frac{4}{\pi}\left[\frac{\cos\Omega t}{1^2} + \frac{\cos 3\Omega t}{3^2} + \frac{\cos 5\Omega t}{5^2} + ..+\right]. \quad (4.16)$$

We wish to examine the transient solution (while the forced oscillation is applied, but a long time after it was first applied, i.e. $t >> 1/b = 1/(2\nu)$). Since the system is damped ($\nu \neq 0$), the solution approaches the particular solution as $t \to \infty$. Since the ODE is linear we can use the **Method of undetermined coefficients**. The solution is therefore the superposition

$$y(t) = y_1(t) + y_3(t) + y_5(t) + ...+, \quad y_n(t) = a_n \cos n\Omega t + b_n \sin n\Omega t$$

where each $y_n(t)$, and thus a_n and b_n, is obtained by substituting $y(t)$ into the ODE (setting $m = 1$, $b = 2\nu$):

$$\frac{d^2 y_n(t)}{dt^2} + 2\nu\frac{dy_n(t)}{dt} + \omega_0^2 y_n(t) = \frac{4A}{n^2\pi}\cos n\Omega t \quad (n = 1, 3, 5...) \quad (4.17)$$

ESD in Forced Oscillations

The result is

$$a_n = \frac{4A}{n^2\pi}\left(\frac{\omega_0^2 - n^2\Omega^2}{D_n}\right), \quad b_n = a_n \frac{2\nu\, n\Omega}{\omega_0^2 - n^2\Omega^2}, \quad D_n = (\omega_0^2 - n^2\Omega^2)^2 + (2\nu n\Omega)^2$$

For evaluating the ESD, we use Eq. (4.7), so that

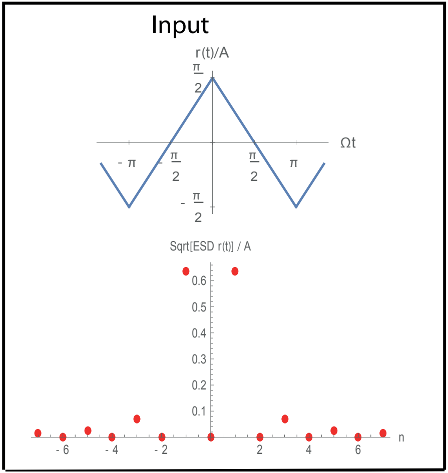
$$\boxed{|c_n|^2 = \frac{1}{4}(a_n^2 + b_n^2) = \frac{4A^2}{n^4\pi^2 D_n}\left\{\frac{1}{2}\left[1 - (-1)^n\right]\right\}.}$$

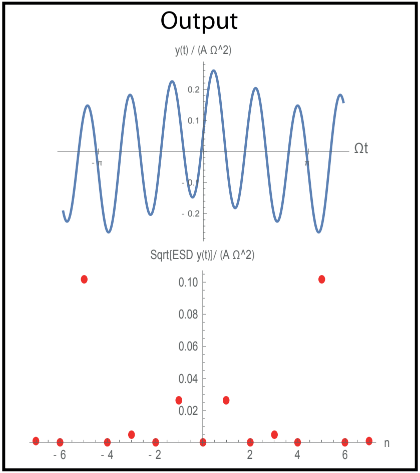
We now examine the results for $\omega_0 = 5\Omega$ and $\nu = \Omega/10$.

Since ν is small, and since $\omega_0 = 5\Omega$, there is a near resonance (small D_n) for $n = 5$. Thus, we expect the solution $y(t)$ to be dominated by y_5, the natural oscillation of the homogeneous system (with frequency ω_0).

In addition, while the input $r(t)$ is purely even, the output is not even - it has small sine components b_n.

ESD in Forced Oscillations





Reverse of Fourier Expansion: identifying closed form of a Fourier Expansion

So far we have taken a known function in closed form, developed a series expansion of it where possible, and in some cases used the expansion to solve problems (e.g. the forced oscillation problem). Sometimes we are presented with an expansion, and we may ask if this series corresponds to a simple closed form associated with basic mathematical functions. One way is to try to look it up on a table. Another way is to try to solve this **inverse problem** by brute force if possible, as shown in the example below.

- Consider the following series valid over $[0, 2\pi]$

  $$\sum_{n=1}^{\infty} \frac{\cos nx}{n} \equiv \lim_{r\to 1} \sum_{n=1}^{\infty} \frac{r^n \cos nx}{n}$$

  where the series on the left is only conditionally convergent. It is convenient to treat the series on the right which is absolutely convergent for $|r| < 1$, and set $r = 1$ after a closed form is obtained.

Identifying closed form of a Fourier Expansion

- Using $\cos nx = (e^{inx} + e^{-inx})/2$ and defining $y = re^{ix}$ and $z = re^{-ix}$ we have

$$\sum_{n=1}^{\infty} \frac{r^n \cos nx}{n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{y^n}{n} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{z^n}{n}$$

- We now use the fact that that the Maclaurin expansion of $\ln(1-\zeta)$ is $-\sum_{n=1}^{\infty} \frac{\zeta^n}{n}$, so that

$$\sum_{n=1}^{\infty} \frac{r^n \cos nx}{n} = -\frac{1}{2}\left[\ln(1-re^{ix}) + \ln(1-re^{-ix})\right].$$

- This is already in closed form. But it can be simplified further by using identities $\ln(ab) = \ln(a) + \ln(b)$, $(1/2)\ln(x) = \ln(x^{1/2})$ and $2[\sin(x/2)]^2 = 1 - \cos x$. Setting $r = 1$ gives,

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\ln\left(2\sin\frac{x}{2}\right), \qquad (0 < x < 2\pi).$$

  The convergence issue in the series expansion for $r = 1$ is connected to singularity in the closed expression for $x \to 0, 2\pi$.

Introduction to Solving PDEs with Fourier Series

- We will concentrate on homogenous linear PDEs with constant coefficients and self-adjoint differential operators. We have already seen that under these conditions, the solution can be decomposed on the basis of the eigenmodes of the differential operator.
- We will start by reviewing the 1D wave equation problem outlined in the introduction to this course. It will be shown how the approach can be extended to multi-dimensional problems. We will see a reconciliation with the well known method of "Separation of Variables".
- In this section we examine problems where the solutions are naturally composed of a Fourier series. We will see that initial profiles with piecewise discontinuities are conveniently treated with Fourier series methods
- In this section we will concentrate on problems with finite sized systems (boundaries that don't extend to infinity), usually where the boundaries extend to a half or full period of a Fourier series.

Homogeneous linear PDE's with constant coefficients

$$\frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \qquad \text{2D wave eqn.}$$

$$0 = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \qquad \text{2D Laplace eqn.}$$

$$0 = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} \qquad \text{3D Laplace eqn.}$$

$$\frac{1}{K^2}\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x^2} \qquad \text{1D heat eqn.}$$

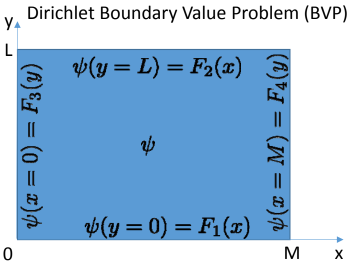
$$-i\frac{2m}{\hbar}\frac{\partial\psi}{\partial t} = \frac{\partial^2\psi}{\partial x^2} \qquad \text{1D hom. linear Schrödinger eqn.}$$

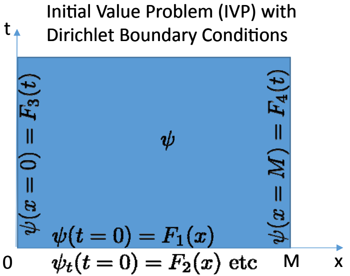
The RHS of all these PDEs is the Laplacian $-\mathcal{D}$. In general the totality of solutions to a PDE is very large. For example, the following are entirely different solutions of the 2D Laplace eqn:

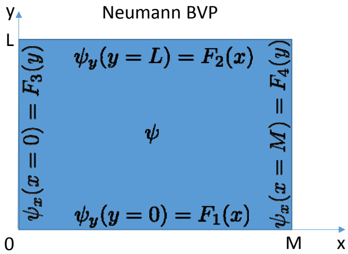
$$\psi = x^2 - y^2,\ \psi = e^x \cos y,\ \psi = \sin x \cosh y,\ \psi = \ln(x^2 + y^2)$$

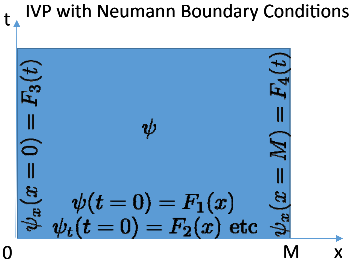
A unique solution will be determined via **boundary conditions**, and for a time dependent problem the **initial conditions**.

Some Common Types Of PDE Boundary Conditions









where $\psi_x = \partial\psi/\partial x$, $\psi_y = \partial\psi/\partial y$,

Notes

The figures on the left are typical of Laplace equation problem. Sometimes the boundary values are set as constant. The boundary value problems can be a combination of Dirichlet and Neumann; such problems are often called **mixed problems**.

For initial value problem figures on the right, clearly the required number of initial conditions at $t = 0$ is determined by the order of the highest time derivative in the PDE, e.g. for a wave equation we require two [$\psi(t = 0)$, and $\partial\psi(t = 0)/\partial t$, and for the heat equation or Schrödinger equation we require only one e.g. $\psi(t = 0)$].

Solving the 1D Wave Equation by constructing an orthogonal basis

We return to our original system of a string of length L attached at both ends. Our goal is to find the solution of the wave equation as outlined first in Eq. (1.3), i.e.

$$\frac{\partial^2\psi(x,t)}{\partial t^2} = u^2\,\frac{\partial^2\psi(x,t)}{\partial x^2},$$

verifying for example the **Dirichlet**-type boundary conditions, Eq. (1.4):

$$\psi(x=0,t)=0, \qquad \text{and} \qquad \psi(x=L,t)=0,$$

as well as the initial, given transverse displacement $\psi_0(x)$ and velocity $\nu_0(x)$, Eq. (1.5):

$$\psi(x,t=0)=\psi_0(x), \qquad \text{and} \qquad \frac{\partial\psi}{\partial t}(x,t=0)=\nu_0(x).$$

- We define the differential operator $\mathcal{D} = -d^2/dx^2$ which acts on the function $\psi(x)$.
- We then attempt to find **eigenfunctions** of $\mathcal{D}$, *i.e.* $\phi(x) \neq 0$ for which there exists a scalar λ such that:

$$\mathcal{D}\phi(x) = -\frac{d^2}{dx^2}\phi(x) = \lambda\phi(x),$$

  together with the boundary conditions $\phi(x=0) = \phi(x=L) = 0$.
- We have already seen that $\mathcal{D}$ only takes on strictly positive eigenvalues, $\lambda > 0$. We identify these positive eigenvalues, by setting $\lambda = \kappa^2$, $\kappa > 0$:

$$\frac{d^2}{dx^2}\phi(x) = -\kappa^2\phi(x).$$

- which has general solution $\phi(x) = A\sin(\kappa x) + B\cos(\kappa x)$, but applying boundary conditions (ψ and thus ϕ have Dirichlet B.C.) give:

$$\phi_k(x) = \sin(\kappa_k x) = \sin(k\frac{\pi}{L}x) \text{ with } k \in \mathbb{N}^\star.$$

  as eigenfunctions of the operator $\mathcal{D}$ wrt. to the eigenvalue $\lambda_k = \kappa_k^2$

Having identified a basis $\{\phi_k(x)\}_{k\in\mathbb{N}^\star}$ of eigenfunctions of $\mathcal{D}$, it becomes straightforward to solve the continuous wave equation:

$$\frac{\partial^2\psi(x,t)}{\partial t^2} = u^2\,\frac{\partial^2\psi(x,t)}{\partial x^2} = -u^2\,\mathcal{D}\,\psi(x,t). \tag{4.18}$$

By decomposing the solution $\psi(x,t)$ on this basis [note time dependence of coefficients $c_k(t)$]:

$$\psi(x,t) = \sum_{k=1}^{+\infty} c_k(t)\phi_k(x),$$

which is then inserted in (4.18), one obtains using $\mathcal{D}\phi_k(x) = \lambda_k\phi_k(x)$, $\lambda_k = \kappa_k^2$:

$$\sum_{k=1}^{+\infty}\frac{d^2c_k(t)}{dt^2}\phi_k(x) = -u^2\sum_{k=1}^{+\infty}c_k(t)\mathcal{D}\phi_k(x) = \sum_{k=1}^{+\infty}-(\kappa_k u)^2c_k(t)\phi_k(x), \tag{4.19}$$

As $\{\phi_k(x)\}$ forms a basis, the coefficients of the left and right hand side of (4.19) must be equal:

$$\frac{d^2}{dt^2}c_k(t) = -(\kappa_k u)^2\,c_k(t) = -\omega_k^2\,c_k(t). \tag{4.20}$$

This is the equation of an harmonic oscillator with frequency $\omega_k = u\kappa_k$, which is identified as the **eigenfrequency** of the eigenmode of vibration $\phi_k(x)$.

The solution to (4.20) is given by:

$$c_k(t) = c_k(0)\cos(\omega_k t) + \frac{\dot{c}_k(0)}{\omega_k}\sin(\omega_k t),$$

where $c_k(0)$ and $\dot{c}_k(0)$ are the **Fourier coefficients** on the basis $\{\phi_k(x)\}$ of the initial displacement velocity fields $\psi_0(x)$ and $\nu_0(x)$ respectively [see initial conds. (1.5)]. For example, to obtain the $\{c_k(0)\}$:

$$\psi_0(x) = \sum_{k=0}^{+\infty} c_k(0)\phi_k(x) \qquad \Longleftrightarrow \qquad c_k(0) = \frac{\langle\phi_k(x)|\psi_0(x)\rangle}{\langle\phi_k(x)|\phi_k(x)\rangle} = \frac{\int_0^L dx\,\phi_k(x)\psi_0(x)}{\int_0^L dx\,\phi_k(x)\phi_k(x)}.$$

Expliciting $\phi_k(x) = \sin(k\frac{\pi}{L}x)$, this same relation takes on the familiar form of a half-range Fourier sine series:

$$\psi_0(x) = \sum_{k=0}^{+\infty} c_k(0)\sin(k\frac{\pi}{L}x) \qquad \Longleftrightarrow \qquad c_k(0) = \frac{2}{L}\int_0^L dx\,\sin(k\frac{\pi}{L}x)\psi_0(x).$$

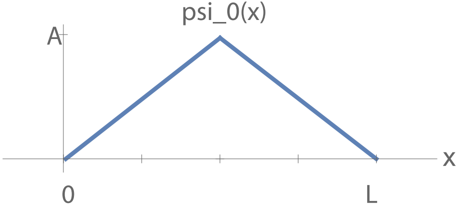
The **general solution** to (4.18) can now be written as a **superposition of eigenmodes of vibrations**:

$$\boxed{\psi(x,t) = \sum_{k=1}^{+\infty}\left[c_k(0)\cos(\omega_k t) + \frac{\dot{c}_k(0)}{\omega_k}\sin(\omega_k t)\right]\sin(k\frac{\pi}{L}x) \quad \text{where,} \quad \omega_k = u\frac{k\pi}{L}}$$

Each eigenmode of vibration $c_k(t)\phi_k(x)$ is a standing wave with frequency ω_k.

Fourier series representation of initial conditions for solution to PDEs

Let us continue to work with the 1D wave equation, but apply it to a problem where the initial distribution is piecewise continuous. The application is that of a plucked guitar string.



The initial distribution of the wave is assumed triangular, i.e. $\psi(t=0,x)=\psi_0(x)$ as above. In addition, we assume that the initial velocity of the wave is zero, i.e. $\nu_0(x)=0$ for any x.

For this **Dirichlet** problem [$\psi(t,x=0)=0$ and $\psi(t,x=L)=0$] the triangular wave must be represented by a sine series (we will see why later):

$$\psi_0(x)=\sum_{k=1}^{\infty}\frac{8A}{k^2\pi^2}\sin\frac{k\pi}{2}\sin\frac{k\pi x}{L}=\sum_{k=1}^{\infty}\frac{8A(-1)^{k-1}}{(2k-1)^2\pi^2}\sin\frac{(2k-1)\pi x}{L}$$

From the earlier results:

$$\psi(x,t) = \sum_{k=1}^{+\infty} \left[c_k(0)\cos(\omega_k t) + \frac{\dot{c}_k(0)}{\omega_k}\sin(\omega_k t) \right] \sin(k\frac{\pi}{L}x) \quad \text{where,} \quad \omega_k = u\frac{k\pi}{L},$$

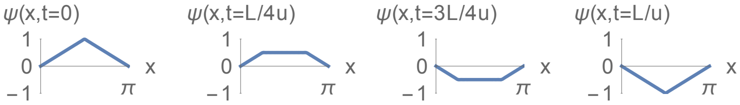
$$\psi_0(x) = \sum_{k=1}^{+\infty} c_k(0)\sin(k\frac{\pi}{L}x) \quad \text{and} \quad \psi_0(x) = \sum_{k=1}^{\infty} \frac{8A}{k^2\pi^2}\sin\frac{k\pi}{2}\sin\frac{k\pi x}{L}$$

giving $c_k(0) = [8A/(k^2\pi^2)]\sin(k\pi/2)$. Also, $\nu_0(x) = 0$ yields $\dot{c}_k(0) = 0$:

$$\psi(x,t) = \sum_{k=1}^{+\infty} \left[\frac{8A}{k^2\pi^2}\sin\frac{k\pi}{2}\cos(\omega_k t) \right] \sin(k\frac{\pi}{L}x) \quad \text{or}$$

$$\psi(x,t) = \sum_{k=1}^{+\infty} \left[\frac{8A(-1)^{k-1}}{(2k-1)^2\pi^2}\cos\left((2k-1)\frac{\pi}{L}ut\right) \right] \sin\left((2k-1)\frac{\pi}{L}x\right).$$

Result is a standing wave. Limiting to 20 terms in series, taking $A = 1$ and $L = \pi$:



Notes

Generalised solution for piecewise continuous initial conditions, or boundary conditions
A word of caution when applying boundary and initial conditions.

- The triangular wave for the initial distribution $\psi(t = 0, x) = \psi_0(x)$ yields a singularity in the wave equation **locally**, since $\partial^2\psi_0/\partial x^2$ does not exist at $x = L/2$. As seen, this differential singularity propagates into the solution domain.
- This singularity remains in the second derivative of the expansion. Term by term differentiation of this expansion can be undertaken (derivatives exist). But, the series representation of $\partial^2\psi_0/\partial x^2$ does not converge at $x = L/2$ for. Clearly, the following type of series does not converge:

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k^0}$$

- However, providing the boundary and initial conditions of ψ are piecewise continuous (e.g. at $x = L/2$), a solution exists everywhere except locally where the derivatives of ψ are singular. Such solutions are known as **generalised solutions**.

Multi-dimensional problems

- All the higher dimensional PDEs listed at the start of this section involved a Laplacian written in cartesian coordinates.
- We therefore expect solutions that are sinusoidal in multi-dimensions.
- We now try to write down a Fourier series of such a solution.

First Fourier decompose the solution $\psi(x, y, z, t)$ in x over the range $[-L_x, L_x]$

$$\psi(x, y, z, t) = \sum_{k=-\infty}^{\infty} c_k(y, z, t) \exp\left[\frac{ik\pi x}{L_x}\right]. \tag{4.21}$$

Applying $\int_{-L_x}^{L_x} dx \exp(-ik\pi x/L_x)$ to both sides, and rearranging:

$$c_k(y, z, t) = \frac{1}{2L_x} \int_{-L_x}^{L_x} dx\, \psi(x, y, z, t) \exp\left[-\frac{ik\pi x}{L_x}\right]. \tag{4.22}$$

We next Fourier analyse c_k in y over the interval $[-L_y, L_y]$:

$$c_k(y, z, t) = \sum_{l=-\infty}^{\infty} c_{k,l}(z, t) \exp\left[\frac{il\pi y}{L_y}\right]. \tag{4.23}$$

Applying $\int_{-L_y}^{L_y} dy \exp(-il\pi y/L_y)$ to both sides, and rearranging:

$$c_{k,l}(z, t) = \frac{1}{2L_y} \int_{-L_y}^{L_y} dy\, c_k(y, z, t) \exp\left[-\frac{il\pi y}{L_y}\right].$$

Multi-dimensional problems

$$c_{k,l}(z,t) = \frac{1}{2L_y} \int_{-L_y}^{L_y} dy\, c_k(y,z,t) \exp\left[-\frac{il\pi y}{L_y}\right]$$

Substituting Eq. (4.22) for $c_k(y,z,t)$ yields the double-Fourier coefficient

$$c_{k,l}(z,t) = \frac{1}{4L_xL_y} \int_{-L_x}^{L_x} dx \int_{-L_y}^{L_y} dy \exp\left[-\frac{ik\pi x}{L_x}\right] \exp\left[-\frac{il\pi y}{L_y}\right] \psi(x,y,z,t) \tag{4.24}$$

which enters in the double complex exponential Fourier series (substitute Eq. (4.23) into Eq. (4.21)):

$$\psi(x,y,z,t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_{k,l}(z,t) \exp\left[\frac{ik\pi x}{L_x}\right] \exp\left[\frac{il\pi y}{L_y}\right] \tag{4.25}$$

Clearly this process can be repeated to obtain a triple complex exponential Fourier series:

$$\psi(x,y,z,t) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{k,l,m}(t) \exp\left[\frac{ik\pi x}{L_x}\right] \exp\left[\frac{il\pi y}{L_y}\right] \exp\left[\frac{im\pi z}{L_z}\right] \tag{4.26}$$

with

$$c_{k,l,m}(t) = \frac{1}{8L_xL_yL_z} \int_{-L_x}^{L_x} dx \int_{-L_y}^{L_y} dy \int_{-L_z}^{L_z} dz \exp\left[-\frac{ik\pi x}{L_x}\right] \exp\left[-\frac{il\pi y}{L_y}\right] \exp\left[-\frac{im\pi z}{L_z}\right] \times\psi(x,y,z,t) \tag{4.27}$$

3D Wave eqn with periodicity over $[-L_x, L_x]$, $[-L_y, L_y]$, $[-L_z, L_z]$

$$u^{-2}\frac{\partial^2\psi}{\partial t^2} - \left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) = 0$$

yields,

$$\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \exp\left[\frac{ik\pi x}{L_x}\right]\exp\left[\frac{il\pi y}{L_y}\right]\exp\left[\frac{im\pi z}{L_z}\right]$$
$$\left[u^{-2}\frac{d^2}{dt^2} + \left(\frac{k\pi}{L_x}\right)^2 + \left(\frac{l\pi}{L_y}\right)^2 + \left(\frac{m\pi}{L_z}\right)^2\right] c_{k,l,m}(t) = 0.$$

The solution to this ODE equation is clearly,

$$c_{k,l,m}(t) = a_{k,l,m}(0)\cos(\omega_{k,l,m}t) + b_{k,l,m}(0)\sin(\omega_{k,l,m}t), \;\; \omega_{k,l,m} = u\pi\left[\frac{k^2}{L_x^2} + \frac{l^2}{L_y^2} + \frac{m^2}{L_y^2}\right]^{\frac{1}{2}}, \tag{4.28}$$

where, from Eq. (4.27), $c_{k,l,m}(t=0)$ is related to $\psi_0(x,y,z)$ and $\nu_0(x,y,z)$ as:

$$a_{k,l,m}(0) = \frac{1}{8L_xL_yL_z}\int_{-L_x}^{L_x} dx \int_{-L_y}^{L_y} dy \int_{-L_z}^{L_z} dz \exp\left[-\frac{ik\pi x}{L_x}\right]\exp\left[-\frac{il\pi y}{L_y}\right]\exp\left[-\frac{im\pi z}{L_z}\right] \times\psi_0(x,y,z),$$

$$b_{k,l,m}(0) = \frac{\omega_{k,l,m}^{-1}}{8L_xL_yL_z}\int_{-L_x}^{L_x} dx \int_{-L_y}^{L_y} dy \int_{-L_z}^{L_z} dz \exp\left[-\frac{ik\pi x}{L_x}\right]\exp\left[-\frac{il\pi y}{L_y}\right]\exp\left[-\frac{im\pi z}{L_z}\right] \times\nu_0(x,y,z).$$

Basis decomposition for a PDE solution

We have seen that the solution of the previous PDE is a **superposition of solutions with separated variables**:

$$\psi = \sum_k \sum_l \sum_m \psi_{k,l,m}, \tag{4.29}$$

$$\psi_{k,l,m} = F_{k,l,m}(t) G_k(x) H_l(y) I_m(z). \tag{4.30}$$

- **ψ is a decomposition of orthogonal basis $G_k(x)$, $H_l(y)$,** $I_m(z)$. For the wave equation with boundary conditions placed on a cube (or a rectangle bounded by the intervals $[-L_x, L_x]$, $[-L_y, L_y]$), the orthogonal basis are complex exponentials.
- The multi-dimensional exponential Fourier series for ψ, although correct, is not the most convenient basis for problems with Dirichlet or Neumann boundary conditions on the restricted cartesian domain $[0, L_x]$, $[0, L_y]$.
- For such problems with Dirichlet boundaries, ψ becomes a half-range multiple sine series, while for such problems with Neumann boundaries, ψ becomes a half range multiple cosine series. Mixed problems can also be envisaged.

B.C. placement, coordinates, and separation of variables

- **The basis expansions for the solution depends entirely on the geometry of the boundary.** If one puts the **boundary on a circle** for a problem with a 2D Laplacian (e.g. 2D wave eqn, 2D heat equation etc), the most natural way to define the Laplacian is in polar coordinates. Having done this, the basis for the problem are not standard multiple Fourier series. As we will see, the basis in the radial harmonics are related to **Bessel functions**.
- Likewise, for a problem with **boundary conditions placed on a sphere** (e.g. the Schrödinger equation for a hydrogen atom), the Laplacian is naturally written in spherical-polar coordinates, and it can be seen that spherical harmonics are connected with **Legendre** polynomials.
- As accomplished earlier for the 3D wave equation, an approach to solving the PDE is to substitute the appropriate multiple basis expansion into the PDE.
- An alternative way to solve the entire problem is to follow a **3-step** approach where the basis decomposition is selected automatically following our choice of boundary conditions, and corresponding choice of coordinates, and to take the solution in the form of Eqs. (4.29) and (4.30). Multiple Sturm-Liouville problems are established from eigenmode equations associated with each orientation of the differential operators $\mathcal{D}_x = -\partial^2/\partial x^2$, $\mathcal{D}_y$ etc.
- Finally, note that the solution for the PDEs considered in this course have solutions of the form of Eqs. (4.29) and (4.30) only because they are linear, have coefficients compatible with a Sturm-Liouville problem, and due to inherent symmetry are thus separable. Other more complicated PDE's can not solved this way.

Double Fourier Series solution to Wave in a rectangular membrane

$$\frac{\partial^2 \psi}{\partial t^2} = u^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right)$$

Boundary conditions

$$\psi(x = 0, x = L_x) = \psi(y = 0, y = L_y) = 0 \quad \text{(Dirichlet type)}$$

$$\text{or } \psi_x(x = 0, x = L_x) = \psi_y(y = 0, y = L_y) = 0 \quad \text{(Neumann type).}$$

Initial conditions

$$\begin{aligned} \psi(x, y, t = 0) &= \psi_0(x, y) \text{ arbitrary for the time being} \\ \psi_t(x, y, t = 0) &= \nu_0(x, y) \text{ arbitrary for the time being} \end{aligned}$$

we now go about solving the problem in three steps

1. Forming three sets of ODE's for this 3 dimensional PDE (2D in space plus time).
2. Solve ODEs satisfying the boundary conditions
3. Solve PDE satisfing the initial conditions

Step 1: obtaining 3 ODEs

From Eq. (4.30) we have $\psi_{k,l} = F_{k,l}(t)G_k(x)H_l(y)$, so that the wave equation gives

$$G_k H_l (F_{k,l})_{tt} = u^2 F_{k,l}[H_l(G_k)_{xx} + G_k(H_l)_{yy}]$$

where subscript t, x and y denote partial derivatives in those variables. Rearranging gives,

$$\frac{(F_{k,l})_{tt}}{u^2 F_{k,l}} = \frac{H_l(G_k)_{xx} + G_k(H_l)_{yy}}{G_k H_l} = -\kappa_{k,l}^2 \tag{4.31}$$

Now, notice that the RHS of the first equality is independent of time (and the LHS is independent of x and y), so that κ is independent of time. We can thus identify the first of the three sets of ODE's:

$$(F_{k,l})_{tt} + \omega_{k,l}^2 F_{k,l} = 0 \text{ with } \omega_{k,l} = u\kappa_{k,l}. \tag{4.32}$$

Now, $H_l(G_k)_{xx} + G_k(H_l)_{yy} = -\kappa_{k,l}^2 G_k H_l$ which arises from Eq. (4.31) can be rearranged to yield

$$\frac{(G_k)_{xx}}{G_k} = -\frac{(H_l)_{yy} + \kappa_{k,l}^2 H_l}{H_l} = -\delta_k^2.$$

The LHS of the first equality in the above is independent of y, and the RHS of the first equality is independent of x, so that δ_k^2 constant **yields ODE's, each being a Sturm-Liouville problem with associated boundary conditions:**

$$(G_k)_{xx} + \delta_k^2 G_k = 0 \tag{4.33}$$

$$(H_l)_{yy} + \nu_l^2 H_l = 0 \text{ where } \nu_l^2 = \kappa_{k,l}^2 - \delta_k^2 \tag{4.34}$$

Step 2: Solving ODEs satisfying boundary conditions

• The general solution to the spatial ODE in x, Eq. (4.33), is written in terms of the basis over k, and the solution to the spatial ODE in y, (4.34) is written in terms of the basis over l:

$$G_k = A\cos(\delta_k x) + B\sin(\delta_k x) \quad \text{and} \quad H_l = C\cos(\nu_l y) + D\sin(\nu_l y)$$

Choosing a **Dirichlet BVP**, $\psi(x=0)=0$ forces $A=0$, and $\psi(y=0)=0$ forces $C=0$.

• The domain of the problem is over by $0 \leq x \leq L_x$ and $0 \leq y \leq L_y$. For a non-trivial problem, we must have B and D non-zero. Hence, the boundary conditions $\psi(x=L_x)=0$ and $\psi(y=L_y)=0$ respectively yield:

$$\sin(\delta_k L_x) = 0 \text{ or } \delta_k = \frac{\pi k}{L_x}, \quad k \in \mathbb{N}^\star \tag{4.35}$$

$$\sin(\nu_l L_y) = 0 \text{ or } \nu_l = \frac{\pi l}{L_y}, \quad l \in \mathbb{N}^\star \tag{4.36}$$

Without loss of generality we can set $B=D=1$, to give

$$G_k(x)H_l(y) = \sin\frac{\pi k x}{L_x}\,\sin\frac{\pi l y}{L_y} \quad \{k,l\} \in \mathbb{N}^\star$$

Step 2: Solving ODEs satisfying boundary conditions

- The solution to the ODE in t, i.e. Eq. (4.32), is of course easily seen to be of the form:

$$F_{k,l}(t) = M_{k,l}\cos\omega_{k,l}t + N_{k,l}\sin\omega_{k,l}t$$

where $\omega_{k,l} = u\kappa_{k,l}$ and from Eq. (4.34) $\kappa_{k,l}^2 = \nu_l^2 + \delta_k^2$, so that the **eigenfrequency** of vibration associated with eigenvectors G_k and H_l is:

$$\boxed{\omega_{k,l} = u\pi\sqrt{\frac{k^2}{L_x^2} + \frac{l^2}{L_y^2}} \qquad \{k,l\} \in \mathbb{N}^\star.} \tag{4.37}$$

- Then, $\psi_{k,l}(t,x,y) = F_{k,l}(t)G_k(x)H_l(y)$ is thus given by

$$\boxed{\psi_{k,l} = (M_{k,l}\cos\omega_{k,l}t + N_{k,l}\sin\omega_{k,l}t)\sin\frac{\pi kx}{L_x}\,\sin\frac{\pi ly}{L_y}.} \tag{4.38}$$

The coefficients $M_{k,l}$ and $N_{k,l}$ are determined respectively from the initial distributions $\psi(x,y,t=0) = \psi_0(x,y)$ and $\psi_t(x,y,t=0) = \nu_0(x,y)$.

Step 3: Solving PDE satisfying initial conditions

The solution to the PDE is the sum of all possible oscillations (eigenfunctions):

$$\psi(x,y,t) = \sum_k \sum_l \psi_{k,l}(x,y,t)$$

$$\psi(x,y,t) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left(M_{k,l} \cos \omega_{k,l} t + N_{k,l} \sin \omega_{k,l} t\right) \sin \frac{\pi k x}{L_x} \sin \frac{\pi l y}{L_y} \qquad (4.39)$$

Applying the initial conditions, we clearly find:

$$\psi(x,y,t=0) = \psi_0(x,y) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} M_{k,l} \sin \frac{\pi k x}{L_x} \sin \frac{\pi l y}{L_y}$$

$$\left.\frac{\partial \psi}{\partial t}\right|_{t=0} = \nu_0(x,y) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \omega_{k,l} N_{k,l} \sin \frac{\pi k x}{L_x} \sin \frac{\pi l y}{L_y}.$$

Coefficients $M_{k,l}$ and $N_{k,l}$ are found by projection. The result is a **double Fourier sine series** with coefficients which can be defined over **half the period**:

$$M_{k,l} = \frac{4}{L_x L_y} \int_0^{L_x} dx \int_0^{L_y} dy\, \psi_0(x,y) \sin \frac{\pi k x}{L_x} \sin \frac{\pi l y}{L_y}$$

$$N_{k,l} = \frac{4}{L_x L_y \omega_{k,l}} \int_0^{L_x} dx \int_0^{L_y} dy\, \nu_0(x,y) \sin \frac{\pi k x}{L_x} \sin \frac{\pi l y}{L_y}.$$

Example of half range double Fourier sine series representation of a paraboloid

- Consider the paraboloid function

$$\psi_0(x,y) = A\left(\frac{x}{L_x} - \frac{x^2}{L_x^2}\right)\left(\frac{y}{L_y} - \frac{y^2}{L_y^2}\right)$$

The double Fourier sine representation of this paraboloid is (see exercise):

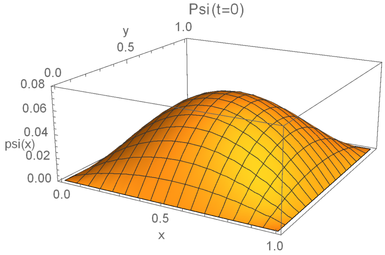
$$\psi_0(x,y) = \left(\frac{2}{\pi}\right)^6 \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{A}{(2k-1)^3(2l-1)^3} \sin\frac{\pi x(2k-1)}{L_x} \sin\frac{\pi y(2l-1)}{L_y}$$

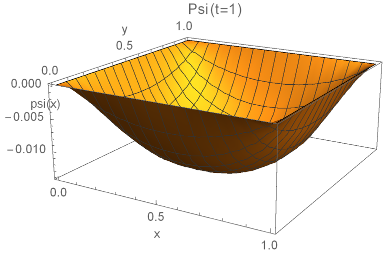
- Given that $\psi_0(x,y)$ is the initial displacement, and choosing the initial velocity to be zero, i.e. $\nu_0 = 0$, the final solution is

$$\psi = \left(\frac{2}{\pi}\right)^6 \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \quad \frac{A}{(2k-1)^3(2l-1)^3} \sin\frac{\pi x(2k-1)}{L_x} \sin\frac{\pi y(2l-1)}{L_y}$$
$$\times \cos\left[u\pi\left(\frac{(2k-1)^2}{L_x^2} + \frac{(2l-1)^2}{L_y^2}\right)^{1/2} t\right]$$

Example of half range double Fourier sine series representation of a paraboloid

In the plots, take $L_x = L_y = 1$, $u = 1$ and $A = 1$:

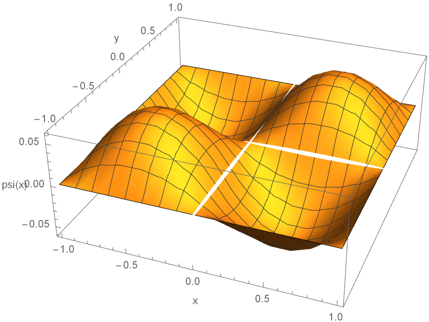




Extension of initial distribution for solving over interval $[-L_x, L_x]$, $[-L_y, L_y]$

As mentioned before, if the boundary conditions of a problem are placed on the perimeter of a rectangle (or box for a 3D problem), we can always solve the problem with a general multiple complex exponential Fourier series. The multiple-complex exponential series defined in Eqs. (4.24) and (4.25) has periodicity over $[-L_x, L_x]$ and $[-L_y, L_y]$.

For the previous 2D wave equation, where the boundary was placed along the interval $[0, L_x]$, and $[0, L_y]$, all that needs to be done is to ensure the initial profile is periodic over interval $[-L_x, L_x]$, $[-L_y, L_y]$. The (2D) complex exponential decomposition of Eqs. (4.26) and (4.27) would give the correct solution with the below initial profile.



Heat equation in one dimension: boundary conditions

The heat equation in one dimension is given by the PDE

$$\frac{\partial \psi}{\partial t} = K^2 \frac{\partial^2 \psi}{\partial x^2},$$

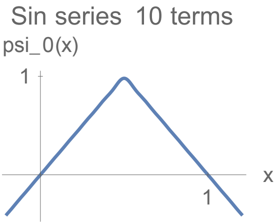
where $\psi(x,t)$ is the temperature along x over time t.

An important related problem is the heat in a metal bar of length L (infinite length bars are considered later in the section on Fourier transforms). Two applications associated with this are:

1. where the two ends of the bar are held at a fixed temperature at all times, e.g. at zero ($\psi(t, x = 0, L) = 0$). This is an initial value problem (IVP) with Dirichlet boundary conditions (BC)
2. where the two ends are insulated, so that the temperature at the ends of the bar can evolve, but the ends of the bar have zero temperature gradient at all times ($\partial\psi(t, x = 0, L)/\partial x = 0$). This is an IVP problem with Neumann BC.

Suitable series representation of initial distribution

- Consider that we choose the initial distribution $\psi_0(x) = \psi(x, t = 0)$ to be a triangular wave over $0 \leq x \leq L$ (choose $L = 1$ in figure).
- As seen in the exercises, we can choose to represent the initial distribution $\psi_0(x)$ as a half range sine, or a half range cosine series. **but, can either be used to solve the physical problems outlined above?**



- For the Dirichlet BC problem ($\psi(t, x = 0, L) = 0$), the initial distribution $\psi(t = 0, x)$ should be chosen to be a sine series (odd function over full period $[-L, L]$). A cosine series would be inconsistent with BC at $t = 0$.
- For the Neumann problem ($\partial\psi(t, x = 0, L)/\partial x = 0$), the initial distribution $\psi(t = 0, x)$ should be chosen to be a cosine series (even function over full period $[-L, L]$). A sine series would be inconsistent with BC at $t = 0$.
- By solving the problem step by step, the appropriate selection is automatic.

One dimensional heat equation with Neumann BC

$$\frac{\partial \psi}{\partial t} = K^2 \frac{\partial^2 \psi}{\partial x^2}$$

Spatial domain $0 \le x \le L$ with boundary conditions

$$\left. \frac{\partial \psi}{\partial x} \right|_{x=0,L} = 0 \quad \text{(Neumann type)}$$

Initial conditions

$$\psi(x, y, t = 0) = \psi_0(x, y) \qquad \text{(later applied to triangular wave).}$$

We now go about solving the problem in three steps

1. Forming two sets of ODE's
2. Solve ODEs satisfying the boundary conditions
3. Solve PDE satisfing the initial conditions

Step 1: obtaining 2 ODEs

From Eq. (4.30) we have $\psi_k(x,t) = F_k(t)G_k(x)$, so that the 1D heat equation gives

$$G_k(F_k)_t = K^2 F_k (G_k)_{xx}$$

where subscript t and x denote partial derivatives in those variables. Rearranging gives,

$$\frac{(F_k)_t}{K^2 F_k} = \frac{(G_k)_{xx}}{G_k} = -\kappa_k^2 \tag{4.40}$$

Now, notice that the RHS is independent of time (and the LHS is independent of x), so that κ is independent of time and x. We can thus identify the two sets of ODE's:

$$\frac{dF_k}{dt} + \omega_k^2 F_k = 0 \quad \text{with} \quad \omega_k = K\kappa_k \tag{4.41}$$

and

$$\frac{d^2 G_k}{dx^2} + \kappa_k^2 G_k = 0 \tag{4.42}$$

Step 2: Solving ODEs satisfying boundary conditions

• The general solution to the spatial ODE equation, Eq. (4.42), is once again

$$G_k = A\cos(\kappa_k x) + B\sin(\kappa_k x) \;\text{ so that }\; \frac{dG_k}{dx} = -A\kappa_k \sin(\kappa_k x) + B\kappa_k \cos(\kappa_k x)$$

The Neumann boundary condition at $x = 0$ forces $B = 0$ (since $dG_k/dx|_{x=0} = 0$) . Thus, G_k is represented by a cosine series. This is consistent with the discussion on the suitable series representation for the initial ($t = 0$) heat distribution.

• The domain of the problem is bounded by $0 \leq x \leq L$. For a non-trivial problem, we must have A non-zero. Hence, the Neumann boundary condition at $x = L$ (i.e. $dG_k/dx|_{x=L} = 0$) yields:

$$\sin(\kappa_k L) = 0 \;\text{ or }\; \kappa_k = \frac{k\pi}{L} \;\text{ with }\; k \in \{0, 1, 2, 3...\}.$$

• The solution to the ODE in time (4.41) is simply $F_k = M_k \exp(-\omega_k^2 t)$. We obtain ψ_k with eigenfrequencies $\omega_k = K\kappa_k = Kk\pi/L$:

$$\boxed{\psi_k(x,t) = M_k \cos\left(\frac{k\pi x}{L}\right)\exp(-\omega_k^2 t) \;\text{ with }\; \omega_k = \frac{Kk\pi}{L} \;\text{ and }\; k \in \{0, 1, 2, 3...\}.} \tag{4.43}$$

Interesting there is a non-zero eigenvector even when the eigenvalue is zero!

Step 3: Solving PDE satisfying initial conditions

The solution to the PDE is the sum of all possible oscillations (eigenfunctions):

$$\begin{aligned}\psi(x,t) &= \sum_k \psi_k(x,t) \\ \psi(x,t) &= \sum_{k=0}^{\infty} M_k \cos\left(\frac{k\pi x}{L}\right) \exp\left[-\left(\frac{Kk\pi}{L}\right)^2 t\right] \qquad (4.44)\end{aligned}$$

At the initial time $t = 0$ we thus have

$$\psi(x,t=0) = \psi_0(x) = \sum_{k=0}^{\infty} M_k \cos\left(\frac{k\pi x}{L}\right),$$

where the coefficient M_k is obtained in terms of the initial distribution $\psi_0(x)$ once again by projection:

$$M_0 = \frac{1}{L}\int_0^L dx\, \psi_0(x), \qquad M_k = \frac{2}{L}\int_0^L dx\, \psi_0(x) \cos\frac{k\pi x}{L} \quad k \in \{1,2,3...\}$$

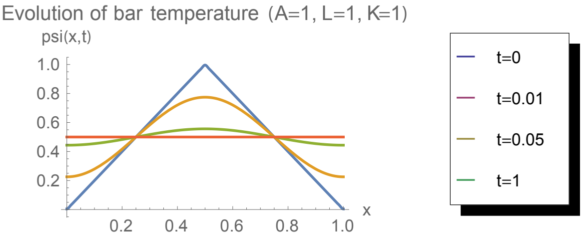
Neumann BC and triangular initial distribution

Taking the cosine series representation for a triangular initial distribution $\psi_0(x)$ with amplitude A, suitable for the Neumann BC for a bar with insulated ends, the Fourier coefficients (exercise) are:

$$M_0 = \frac{A}{2}, \quad M_k = \frac{4A}{k^2\pi^2}\left(2\cos\frac{k\pi}{2} - \cos k\pi - 1\right) \quad k \in \{1, 2, 3...\}$$

This gives a full solution in x and t:

$$\psi = \frac{A}{2} + \sum_{k=1}^{\infty} \frac{4A}{k^2\pi^2}\left(2\cos\frac{k\pi}{2} - \cos k\pi - 1\right)\cos\left(\frac{k\pi x}{L}\right)\exp\left[-\left(\frac{Kk\pi}{L}\right)^2 t\right]$$



Perfect insulated ends of bar prevent energy loss, so that average temperature along bar is conserved!

Fourier-bessel basis associated with 2D wave equation with boundary conditions placed on a circle

We now consider a **2D wave equation** where we place the boundary conditions on a circle $r = R$, and take ψ on this surface to be independent of poloidal angle (choose Dirichlet on $r = R$):

$$\psi(R) = 0 \ \text{ for } t \geq 0 \text{ (Dirichlet type).}$$

Choose initial conditions with poloidal symmetry (relaxed in exercise).

$$\begin{aligned} \psi(r, t = 0) &= \psi_0(r) \\ \psi_t(r, t = 0) &= \nu_0(r) \end{aligned}$$

Since the boundary is circular, we naturally choose to define the 2D Laplacian in terms of polar coordinates. Due to poloidal symmetry in boundary and initial conditions, we have $\psi = \psi(r, t)$. The wave equation is therefore,

$$\frac{\partial^2 \psi}{\partial t^2} = u^2 \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} \right).$$

The problem can be solved by expanding ψ in terms of the Fourier-Bessel basis, and substituting into the the PDE (exercise). Or, we assume the general separable form of Eq. (4.30), and follow the three steps (which can easily be generalised to include non-poloidal symmetry)

1. Forming two sets of ODE's
2. Solve ODEs satisfying the boundary conditions
3. Solve PDE satisfing the initial conditions

Step 1: obtaining 2 ODEs

From Eq. (4.30) we have $\psi_k(r,t) = F_k(t)G_k(r)$, so that the cylindrical wave equation gives

$$G_k(F_k)_{tt} = u^2 F_k \left[(G_k)_{rr} + r^{-1}(G_k)_r\right]$$

where subscript t and r denote partial derivatives in those variables. Rearranging gives,

$$\frac{(F_k)_{tt}}{u^2 F_k} = \frac{1}{G_k}\left[(G_k)_{rr} + \frac{1}{r}(G_k)_r\right] = -\kappa_k^2 \tag{4.45}$$

Now, notice that the RHS is independent of time (and the LHS is independent of r), so that κ is independent of time and r. We can thus identify the ODE's:

$$\frac{d^2 F_{k,l}}{dt^2} + \omega_{k,l}^2 F_{k,l} = 0 \quad \text{with } \omega_{k,l} = u\kappa_{k,l}. \tag{4.46}$$

and

$$\frac{d^2 G_k}{dr^2} + \frac{1}{r}\frac{dG_k}{dr} + \kappa_k^2 G_k = 0 \tag{4.47}$$

The latter equation can be written as **Bessel's equation** by a change of variable $s = \kappa_k r$:

$$\frac{d^2 G_k}{ds^2} + \frac{1}{s}\frac{dG_k}{ds} + G_k = 0 \tag{4.48}$$

Step 2: Solving ODEs satisfying boundary conditions

• The general solution to the Bessel equation, Eq. (4.48), is (see ODE section) the sum of Bessel functions of the first kind J_0 and second kind Y_0.

$$G_k = AJ_0(r\kappa_k) + BY_0(r\kappa_k)$$

Now, $Y_0(r\kappa_k \to 0) \to \infty$, so that for a physical solution on the axis of symmetry, we require $B = 0$.

• The domain of the problem is $0 \le r \le R$. For a non-trivial problem, we must have A non-zero. Hence, the boundary condition $\psi(r = R) = 0$ yields:

$$J_0(\kappa_k R) = 0 \quad \text{or} \quad \kappa_k = \frac{\alpha_k}{R},$$

where the infinite set α_k satisfies $J_0(\alpha_k) = 0$, with $k \in \mathbb{N}^\star$.
Thus the functions

$$G_k(r) = J_0\left(\frac{\alpha_k r}{R}\right) \qquad k \in \mathbb{N}^\star$$

are solutions of the ODE (4.47) and are zero on the boundary circle $r = R$.

• The solution to the ODE in time (4.46) is simply $F_k = M_k \cos\omega_k t + N_k \sin\omega_k t$. Given that $\omega_k = u\kappa_k$, and using $\psi_k = F_k G_k$, we obtain the solution ψ_k with eigenfrequencies $\alpha_k u/R$:

$$\psi_k(r,t) = \left[M_k \cos\frac{\alpha_k ut}{R} + N_k \sin\frac{\alpha_k ut}{R}\right] J_0\left(\frac{\alpha_k r}{R}\right) \qquad (4.49)$$

Step 3: Solving PDE satisfying initial conditions

The solution to the PDE is the sum of all possible oscillations. This generates a **Fourier-Bessel series** (leaving aside problems of convergence and uniqueness):

$$\begin{aligned}\psi(r,t) &= \sum_k \psi_k(r,t) \\ \psi(r,t) &= \sum_{k=1}^{\infty}\left[M_k \cos\frac{\alpha_k u t}{R} + N_k \sin\frac{\alpha_k u t}{R}\right] J_0\left(\frac{\alpha_k r}{R}\right) \qquad (4.50)\end{aligned}$$

Considering the initial conditions, we clearly find:

$$\begin{aligned}\psi(r,t=0) &= \psi_0(r) = \sum_{k=1}^{\infty} M_k J_0\left(\frac{\alpha_k r}{R}\right) \\ \left.\frac{\partial\psi}{\partial t}\right|_{t=0} &= \nu_0(r) = \sum_{k=1}^{\infty}\frac{\alpha_k u}{R} N_k J_0\left(\frac{\alpha_k r}{R}\right).\end{aligned}$$

Clearly, M_k and $N_k\alpha_k u/R$ are the coefficients of a Bessel series (see ODE section). These coefficients are easily calculated in terms of the initial distributions $\psi_0(r)$ and $\nu_0(r)$ by projection:

$$\begin{aligned}M_k &= \frac{2}{R^2[J_1(\alpha_k r/R)]^2}\int_0^R dr\,\psi_0(r) r J_0\left(\frac{\alpha_k r}{R}\right) \quad k \in \mathbb{N}^\star \\ N_k &= \left(\frac{R}{\alpha_k u}\right)\frac{2}{R^2[J_1(\alpha_k r/R)]^2}\int_0^R dr\,\nu_0(r) r J_0\left(\frac{\alpha_k r}{R}\right).\end{aligned}$$

Notes

The development of the coefficient M_k depends on the differentiability of $\psi_0(r)$. In general, for a **generalised solution**, we require that $\psi_0(r)$ is once again piecewise continuous.
As seen earlier in this course, the Bessel-Fourier series is an orthogonal expansion, so that for the case at hand,

$$\int_0^R dr\, r J_0(\alpha_k r/R) J_0(\alpha_l r/R) = 0 \quad k, l \in \mathbb{N}^\star \quad \text{and} \quad k \neq l$$

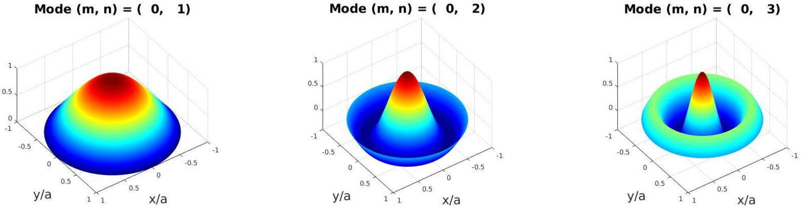
where $J_0(\alpha_k) = 0$.
Moreover for the special case $k = l$ we have

$$\int_0^R dr\, r \left[J_0(\alpha_k r/R)\right]^2 = \frac{R^2}{2} \left[J_1(\alpha_k r/R)\right]^2 \quad k \in \mathbb{N}^\star$$

where $J_n(x)$ satisfies the ODE

$$x^2 \frac{dJ_n(x)}{dx^2} + x \frac{dJ_n(x)}{dx} + (x^2 - n^2) J_n(x).$$

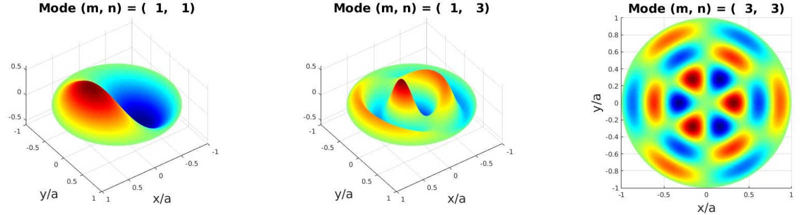
2D wave equation: poloidally symmetric modes of vibration in a circular disc



Examples of eigenmodes of vibration with different radial mode numbers n. Modes have angular symmetry ($m = 0$)

2D wave equation: modes of vibration in a circular disk that break poloidal symmetry

Problem treated in the exercises:



Examples of eigenmodes of vibration of a disc with different poloidal and radial mode numbers m and n respectively.